

Analisi Matematica II
Integrali doppi e tripli

Esercizio 1. Calcolare i seguenti integrali doppi

- (1) $\iint_D x^3 y \, dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, -1 \leq y \leq 0\}$,
- (2) $\iint_D \frac{x^2}{(y+1)^2} \, dx dy$, dove $D = [0, 1] \times [1, 2]$,
- (3) $\iint_D \sin x \cos y \, dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \pi, 0 \leq y \leq \pi\}$,
- (4) $\iint_D e^{x+y} \, dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \pi\}$,
- (5) $\iint_D \frac{1}{1-x-y+xy} \, dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : -\frac{1}{2} \leq x \leq \frac{1}{2}, 0 \leq y \leq \frac{1}{2}\}$,
- (6) $\iint_D \frac{1}{(1+x+y)^2} \, dx dy$, dove D è il triangolo di vertici $(2, 1), (1, 2), (1, 0)$,
- (7) $\iint_D x(y + \sin(\pi y)) \, dx dy$, dove D è il triangolo di vertici $(2, 1), (1, 2), (1, 0)$,
- (8) $\iint_D x^2 e^{xy} \, dx dy$, dove D è il triangolo di vertici $(0, 0), (1, 0), (1, 1)$,
- (9) $\iint_D x^2 e^{xy} \, dx dy$, dove D è il triangolo di vertici $(0, 0), (-1, 2), (1, 2)$,
- (10) $\iint_D e^{y^2} \, dx dy$, dove D è il triangolo di vertici $(0, 0), (-1, 2), (1, 2)$,
- (11) $\iint_D xy \, dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 0, -x^2 \leq y \leq 1+x\}$,
- (12) $\iint_D (x^2 + y) \, dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, 0 \leq y \leq 1-x^2\}$,
- (13) $\iint_D e^{-(x-y)} \, dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : x \geq 0, x \leq y \leq 2-2x\}$,
- (14) $\iint_D x^2 \, dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2\pi, \sin x \leq y \leq 1 + \sin x\}$,
- (15) $\iint_D y^2 \, dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2\pi, \sin x \leq y \leq 1 + \sin x\}$,
- (16) $\iint_D \sin(x+y) \, dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq x\}$,
- (17) $\iint_D y \cos x \, dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y^2 + \pi, -\sqrt{\pi} \leq y \leq 2\sqrt{\pi}\}$,

Esercizio 2. Calcolare i seguenti integrali doppi

- (1) $\iint_D \frac{|x|}{\sqrt{1+y^2}} \, dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, 0 \leq y \leq 3\}$,

- (2) $\iint_D \frac{1}{(x-y+6)^2} dx dy$, dove $D = [0, 1] \times [1, 2]$,
- (3) $\iint_D \frac{1-3x^2}{(x-y+6)^2} dx dy$, dove $D = [0, 1] \times [1, 2]$,
- (4) $\iint_D \frac{1-3x^2}{x-y+6} dx dy$, dove $D = [0, 1] \times [1, 2]$,
- (5) $\iint_D \frac{1}{2x+y+1} dx dy$, dove $D = [1, 2] \times [0, 1]$,
- (6) $\iint_D \frac{x}{2x+y+1} dx dy$, dove $D = [1, 2] \times [0, 1]$,
- (7) $\iint_D \frac{1}{y^2+2x+1} dx dy$, dove $D = [1, 2] \times [0, 1]$,
- (8) $\iint_D \frac{x}{y^2+2x+1} dx dy$, dove $D = [1, 2] \times [0, 1]$,
- (9) $\iint_D \log(x+y+5) dx dy$, dove D è il triangolo di vertici $(-2, 1), (1, 1), (-2, -2)$,
- (10) $\iint_D \frac{y^2+1}{y^2+2x+1} dx dy$, dove D è il triangolo di vertici $(0, 0), (1, 1), (1, -1)$,
- (11) $\iint_D \frac{1}{(1+x+y)^2} dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 2\sqrt{x}\}$,
- (12) $\iint_D x(y + \sin(\pi y)) dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 2\sqrt{x}\}$,
- (13) $\iint_D xy dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : -1 + \cos y \leq x \leq 1 - \cos y, 0 \leq y \leq 2\pi\}$,
- (14) $\iint_D (x-1)|y| dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : 1 \leq |x| + |y| \leq 2\}$,
- (15) $\int_0^1 \left(\int_x^{x^2} \frac{1}{2x+y+1} dy \right) dx$,
- (16) $\int_0^1 \left(\int_x^{x^2} \frac{x}{2x+y+1} dy \right) dx$,
- (17) $\int_0^1 \left(\int_x^{\sqrt{x}} \frac{1}{y^2+2x+1} dy \right) dx$,
- (18) $\iint_D \frac{x^2}{(y+1)^2} dx dy$, dove D è l'insieme limitato dalle curve $y = x^2$ e $y = 2x + 3$,
- (19) $\iint_D \frac{1}{2x+y+2} dx dy$, dove D è l'insieme del piano limitato dalle curve $y = x^2$ e $y = 2x + 3$,
- (20) $\iint_D \frac{1}{(2x-y+8)^2} dx dy$, dove D è l'insieme limitato dalle curve $y = x^2$ e $y = 2x + 3$,
- (21) $\iint_D \frac{1-3x^2}{(2x-y+8)^2} dx dy$, dove D è l'insieme limitato dalle curve $y = x^2$ e $y = 2x + 3$,

$$(22) \iint_D \frac{1 - 3x^2}{2x - y + 8} dx dy, \text{ dove } D \text{ è l'insieme limitato dalle curve } y = x^2 \text{ e } y = 2x + 3,$$

Esercizio 3. Calcolare i seguenti integrali doppi

- (1) $\iint_D \sqrt{x^2 + y^2} dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 3\}$,
- (2) $\iint_D 2xy dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9, y \geq x, x \leq 0\}$,
- (3) $\iint_D \left| \frac{x}{y} \right| dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 16, 0 \leq x \leq |y|\}$,
- (4) $\iint_D \frac{1}{x} dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4, -x \leq y \leq x\}$,
- (5) $\iint_D (x - y)^2 dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1, |x| \leq 2, |y| \leq 2\}$,
- (6) $\int_0^4 \left(\int_{-\sqrt{4x-x^2}}^{\sqrt{4x-x^2}} \sqrt{y^2 + x^2} dy \right) dx$,
- (7) $\iint_D \frac{x^2}{x^2 + y^2} dx dy$, dove D è l'insieme limitato dalle curve $x^2 + y^2 = 1$ e $x^2 + y^2 = 9$,
- (8) $\iint_D \frac{x^2}{x^2 + y^2} dx dy$, dove D è l'insieme limitato da $x^2 + (y - 2)^2 = 4$ ed esterno a $x^2 + y^2 = 4$,
- (9) $\iint_D |xy| dx dy$, dove D è l'insieme limitato dalle curve $x^2 + y^2 = 1$ e $x^2 + y^2 = 9$,
- (10) $\iint_D |xy| dx dy$, dove D insieme limitato dalla circonferenza di centro $(0, 3)$ e raggio 3,
- (11) $\iint_D |xy| dx dy$, dove D è l'insieme limitato da $x^2 + (y - 2)^2 = 4$ ed esterno a $x^2 + y^2 = 4$,
- (12) $\iint_D \frac{|xy|e^{x^2+y^2}}{x^2 + y^2} dx dy$, dove D è l'insieme limitato dalle curve $x^2 + y^2 = 1$ e $x^2 + y^2 = 9$,
- (13) $\iint_D \frac{|xy|e^{x^2+y^2}}{x^2 + y^2} dx dy$, dove D è l'insieme limitato da $x^2 + (y - 2)^2 = 4$ ed esterno a $x^2 + y^2 = 4$.
- (14) $\iint_D x^3 y dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : x^2 + \frac{1}{4}y^2 \leq 2, y \geq 2x\}$,
- (15) $\iint_D |y| dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : x^2 + \frac{1}{4}y^2 \leq 2, y \geq 2x\}$,
- (16) $\iint_D (x - y) \sin(x + y) dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : |x| \leq y \leq 2\}$,
- (17) $\iint_D (x^2 - y^2) \sin(x + y) dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : |y| \leq x \leq 2 - |y|\}$,
- (18) $\iint_D \sqrt{x + y} dx dy$, dove D è il parallelogramma di vertici $(0, 0), (1, 2), (4, 3), (3, 1)$.

Esercizio 4. Calcolare l'area delle seguenti regioni

- (1) $D = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, x^2 \leq y \leq 1\}$,
- (2) $D = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 2, x^2 \leq y \leq x + 2\}$,
- (3) $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2, 0 \leq y \leq 2, xy \leq 1\}$,
- (4) $D = \left\{ (x, y) \in \mathbb{R}^2 : \frac{1}{4}y^2 \leq x \leq 2y, 0 \leq y \leq 8 \right\}$,
- (5) $D = \{(x, y) \in \mathbb{R}^2 : (x^2 + y^2)^3 \leq 16x^2\}$,
- (6) $D = \{(x, y) \in \mathbb{R}^2 : 9 \leq x^2 + y^2 \leq 8y\}$,
- (7) $D = \left\{ (x, y) \in \mathbb{R}^2 : x^2 + \frac{1}{4}y^2 \leq 2, y \geq 2x \right\}$.

Esercizio 5. Calcolare i seguenti integrali tripli

- (1) $\iiint_D \frac{x^2 y}{z} dx dy dz$, dove $D = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1, 0 \leq y \leq 1, 1 \leq z \leq 2\}$,
- (2) $\iiint_D x \sin^2 y \cos z dx dy dz$, dove $D = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1, 0 \leq y \leq \pi, 0 \leq z \leq \frac{\pi}{2}\}$,
- (3) $\iiint_D 2(x + y + z) dx dy dz$, dove $D = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$,
- (4) $\iiint_D (x + y + z) dx dy dz$, dove $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq z^2, -1 \leq z \leq 2\}$,
- (5) $\iiint_D (x^3 + y^3 + z^3) dx dy dz$, dove $D = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x, 0 \leq y, 0 \leq z, x + y + z \leq 1\}$,
- (6) $\iiint_D \frac{1}{\sqrt{1 - z^2}} dx dy dz$, dove $D = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq y \leq z \leq \frac{1}{2}\}$,
- (7) $\iiint_D xyz^2 dx dy dz$, dove $D = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1, -x \leq z \leq x, x + z \leq y \leq 4\}$,
- (8) $\iiint_D (x^2 + y^2 + z^2) dx dy dz$, dove $D = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 2(1 - x - y)\}$,
- (9) $\iiint_D \frac{x}{\sqrt{x^2 + y^2}} dx dy dz$, dove $D = \{(x, y, z) : 0 \leq z \leq 1; 0 \leq y \leq x; \frac{1}{16} \leq x^2 + y^2 \leq 1\}$,
- (10) $\iiint_D \frac{2z}{(x^2 + y^2)^{3/2}} \cos\left(\frac{\arctg(y/x)}{\sqrt{x^2 + y^2}}\right) dx dy dz$, dove $D = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq z \leq 1; 0 \leq y \leq x; \frac{1}{16} \leq x^2 + y^2 \leq 1\}$,
- (11) $\iiint_D x^2 dx dy dz$, dove $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$,
- (12) $\iiint_D (y - 3)^2 dx dy dz$, dove $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4\}$,
- (13) $\iiint_D x^2 y dx dy dz$, dove $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1, 0 \leq y \leq 1\}$,

- (14) $\iiint_D \left(e^z + \frac{2y}{1+x^2+y^2} + x - 1 \right) dx dy dz$, dove $D = \{(x, y, z) \in \mathbb{R}^3 : (x-1)^2 + y^2 + z^2 \leq 1\}$,
- (15) $\iiint_D \left(e^z + z(x-1)^2 + \frac{2z}{1+(x-1)^2 + (y/2)^2 + z^2} \right) dx dy dz$, dove $D = \{(x, y, z) \in \mathbb{R}^3 : (x-1)^2 + \frac{1}{4}y^2 + z^2 \leq 1\}$.

Esercizio 6. Calcolare il volume delle seguenti regioni

- (1) $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4, z \geq 1\}$,
- (2) $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 4y^2 - 9z^2 \leq 1, -1 \leq z \leq 1\}$,
- (3) $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4, x^2 + y^2 \leq 1\}$,
- (4) $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, 0 \leq z \leq x^2 + y^2\}$,
- (5) $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, -1 \leq z \leq x^2 - y^2\}$,
- (6) $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 4y^2 + 9z^2 \leq 1\}$,
- (7) $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1, z \geq 0, x^2 + y^2 + (z+1)^2 \geq 2\}$.

Esercizio 7. Calcolare i seguenti integrali impropri

- (1) $\iint_D \frac{\operatorname{arctg} y^2}{y} dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : \sqrt{|x|} \leq y \leq 1, |x| \leq 1\}$,
- (2) $\iint_D \frac{\operatorname{arctg} y}{y^2} dx dy$, dove $D = \{(x, y) \in \mathbb{R}^2 : \sqrt{|x|} \leq y \leq 1, |x| \leq 1\}$,
- (3) $\int_0^1 \left(\int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{(y^2 + x)^2} dy \right) dx$,
- (4) $\iint_D \frac{(x+2)|y|}{y^2 + 2x + 1} dx dy$, dove D è l'insieme limitato dalle curve $x = \frac{1}{2}(y^2 - 1)$ e $x = 0$,
- (5) $\iint_D \frac{(x+2)y}{y^2 + 2x} dx dy$, dove D è l'insieme limitato dalle curve $y = \sqrt{x}$ e $y = x$,
- (6) $\int_0^1 \left(\int_x^{\sqrt{2x-x^2}} \frac{1}{\sqrt{y^2 + x^2}} dy \right) dx$.
- (7) $\iint_D \frac{x^2}{x^2 + y^2} dx dy$, dove D è l'insieme limitato dalla circonferenza di centro $(0, 3)$ e raggio 3,
- (8) $\iint_D \frac{|xy|e^{x^2+y^2}}{x^2 + y^2} dx dy$, dove D è l'insieme limitato dalla circonferenza di centro $(0, 3)$ e raggio 3,
- (9) $\iint_D \sqrt{\frac{x^2}{|y|}} dx dy$, dove D è l'insieme limitato da $x^2 + y^2 = 4$ ed esterno a $x^2 + (y-1)^2 = 1$,
- (10) $\iint_D \left| \frac{x}{y} \right| dx dy$, dove D è l'insieme limitato dalla curva $\frac{1}{4}x^2 + y^2 = 4$.
- (11) $\iint_D e^{\frac{x+y}{x-y}} dx dy$, dove D è il triangolo di vertici $(0, 0)$, $(1, 0)$, $(0, -1)$,

- (12) $\iint_D \log \frac{x+y}{x-y} dx dy$, dove D è il triangolo di vertici $(0,0)$, $(1,0)$, $(1,1)$,
- (13) $\iint_D \frac{x}{\sqrt{|y|}} dx dy$, dove D è il triangolo di vertici $(-2,2)$, $(2,2)$, $(-2,-2)$,
- (14) $\iint_D xy dx dy$, dove $D = \{(x,y) \in \mathbb{R}^2 : x \geq 0, 0 \leq y \leq e^{-x}\}$,
- (15) $\iint_D \frac{x^2 y^2}{(1+x^6)(1+y^6)} dx dy$, dove $D = \mathbb{R}^2$,
- (16) $\iint_D \frac{1}{(2+x^2+y^2)\sqrt{x^2+y^2}} dx dy$, dove $D = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$,
- (17) $\iint_D \frac{x^2 y^2 e^{-x^2-y^2}}{x^2+y^2} dx dy$, dove $D = \mathbb{R}^2$,
- (18) $\iiint_D \frac{dx dy dz}{xz^2}$, dove $D = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq x \leq 1, |y| \leq x, 1 \leq z \leq \frac{4}{4x-y^2}\}$.
- (19) $\iiint_D \frac{z}{\sqrt{1-x^2-4y^2-z^2}} dx dy dz$, dove $D = \{(x,y,z) \in \mathbb{R}^3 : x^2+4y^2+z^2 \leq 1, z \geq 0\}$.
- (20) $\iiint_D \frac{x^2}{1+z^6} dx dy dz$, dove $D = \{(x,y,z) \in \mathbb{R}^3 : z \geq x^2+y^2\}$.
- (21) $\iiint_D \frac{x^2}{(x^2+y^2+z^2)^4} dx dy dz$, dove $D = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \geq 1\}$.

Analisi Matematica II
Integrali doppi e tripli (svolgimenti)

Svolgimento esercizio 1

- (1) Si ha $\iint_D x^3 y \, dx dy = \int_0^1 x^3 \, dx \cdot \int_{-1}^0 y \, dy = \left[\frac{x^4}{4}\right]_0^1 \cdot \left[\frac{1}{2} y^2\right]_{-1}^0 = -\frac{1}{8}$.
- (2) Si ha $\iint_D \frac{x^2}{(y+1)^2} \, dx dy = \int_0^1 x^2 \, dx \cdot \int_1^2 \frac{dy}{(y+1)^2} = \left[\frac{x^3}{3}\right]_0^1 \cdot \left[-\frac{1}{y+1}\right]_1^2 = \frac{1}{18}$.
- (3) Si ha $\iint_D \sin x \cos y \, dx dy = \int_0^\pi \sin x \, dx \cdot \int_0^\pi \cos y \, dy = [-\cos x]_0^\pi \cdot [\sin y]_0^\pi = 0$.
- (4) Si ha $\iint_D e^{x+y} \, dx dy = \int_0^{\frac{\pi}{2}} e^x \, dx \cdot \int_0^\pi e^y \, dy = [e^x]_0^{\frac{\pi}{2}} \cdot [e^y]_0^\pi = (e^{\frac{\pi}{2}} - 1)(e^\pi - 1)$.
- (5) Si ha $\iint_D \frac{1}{1-x-y+xy} \, dx dy = \iint_D \frac{1}{(1-x)(1-y)} \, dx dy = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{1-x} \cdot \int_0^{\frac{1}{2}} \frac{dy}{1-y} = [-\log|1-x|]_{-\frac{1}{2}}^{\frac{1}{2}} \cdot [-\log|1-y|]_0^{\frac{1}{2}} = \log 2 \cdot \log 3$.
- (6) Si ha $\iint_D \frac{1}{(1+x+y)^2} \, dx dy = \int_1^2 \left[-\frac{1}{1+x+y}\right]_{y=x-1}^{y=3-x} dx = \int_1^2 \left(\frac{1}{2x} - \frac{1}{4}\right) dx = \left[\frac{1}{2} \log x - \frac{1}{4} x\right]_1^2 = \frac{1}{2} \log 2 - \frac{1}{4}$.
- (7) Si ha $\iint_D x(y + \sin(\pi y)) \, dx dy = \int_1^2 x \left[\frac{1}{2} y^2 - \frac{1}{\pi} \cos(\pi y)\right]_{y=x-1}^{y=3-x} dx = \int_1^2 x \left(\frac{1}{2} (3-x)^2 - \frac{1}{\pi} \cos(3\pi - \pi x) - \frac{1}{2} (x-1)^2 + \frac{1}{\pi} \cos(\pi x - \pi)\right) dx = 2 \int_1^2 (2x - x^2) dx = 2 \left[x^2 - \frac{1}{3} x^3\right]_1^2 = \frac{4}{3}$.
- (8) Si ha $\iint_D x^2 e^{xy} \, dx dy = \int_0^1 \left(\int_0^x x^2 e^{xy} dy\right) dx \stackrel{(a)}{=} \int_0^1 x \left(\int_0^{x^2} e^t dt\right) dx = \int_0^1 x [e^t]_0^{x^2} dx = \int_0^1 x (e^{x^2} - 1) dx = \left[\frac{1}{2} e^{x^2} - \frac{1}{2} x^2\right]_0^1 = \frac{e}{2} - 1$, dove in (a) si è usato il cambio di variabile $xy = t \implies xdy = dt$.
- (9) Si ha $\iint_D x^2 e^{xy} \, dx dy = 2 \int_0^1 \left(\int_{2x}^2 x^2 e^{xy} dy\right) dx \stackrel{(a)}{=} 2 \int_0^1 x \left(\int_{2x^2}^{2x} e^t dt\right) dx = 2 \int_0^1 x [e^t]_{2x^2}^{2x} dx = 2 \int_0^1 x (e^{2x} - e^{2x^2}) dx = 2 \left[\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} - \frac{1}{4} e^{2x^2}\right]_0^1 = 1$, dove in (a) si è usato il cambio di variabile $xy = t \implies xdy = dt$.
- (10) Si ha $\iint_D e^{y^2} \, dx dy = \int_0^2 e^{y^2} [x]_{-y/2}^{y/2} dy = \int_0^2 y e^{y^2} dy \stackrel{(a)}{=} \frac{1}{2} \int_0^4 e^t dt = \frac{1}{2} [e^t]_0^4 = \frac{1}{2} (e^4 - 1)$, dove in (a) si è usato il cambio di variabile $y^2 = t \implies 2ydy = dt$.
- (11) Si ha $\iint_D xy \, dx dy = \int_{-1}^0 \left(\int_{-x^2}^{1+x} xy dy\right) dx = \frac{1}{2} \int_{-1}^0 x [y^2]_{y=-x^2}^{y=1+x} dx = \frac{1}{2} \int_{-1}^0 (x + 2x^2 + x^3 - x^5) dx = \frac{1}{2} \left[\frac{1}{2} x^2 + \frac{2}{3} x^3 + \frac{1}{4} x^4 - \frac{1}{6} x^6\right]_{-1}^0 = -\frac{1}{8}$.
- (12) Si ha $\iint_D (x^2 + y) \, dx dy = \int_{-1}^1 \left(\int_0^{1-x^2} (x^2 + y) dy\right) dx = \int_{-1}^1 \left[x^2 y + \frac{1}{2} y^2\right]_{y=0}^{y=1-x^2} dx = \int_{-1}^1 (1 - x^4) dx = \left[x - \frac{1}{5} x^5\right]_{-1}^1 = \frac{4}{5}$.
- (13) Si ha $\iint_D e^{-(x-y)} \, dx dy = \int_0^{2/3} e^{-x} \left(\int_x^{2-2x} e^y dy\right) dx = \int_0^{2/3} e^{-x} [e^y]_{y=x}^{y=2-2x} dx = \int_0^{2/3} (e^{2-3x} - 1) dx = \left[-\frac{1}{3} e^{2-3x} - x\right]_0^{2/3} = \frac{1}{3} e^2 - 1$.
- (14) Si ha $\iint_D x^2 \, dx dy = \int_0^{2\pi} x^2 \left(\int_{\sin x}^{1+\sin x} dy\right) dx = \int_0^{2\pi} x^2 dx = \left[\frac{1}{3} x^3\right]_0^{2\pi} = \frac{8}{3} \pi^3$.
- (15) Si ha $\iint_D y^2 \, dx dy = \int_0^{2\pi} \left[\frac{1}{3} y^3\right]_{\sin x}^{1+\sin x} dx = \frac{1}{3} \int_0^{2\pi} (1 + 3 \sin x + 3 \sin^2 x) dx = \frac{1}{3} \left[x - 3 \cos x + \frac{3}{2} x - \frac{3}{4} \sin 2x\right]_0^{2\pi} = \frac{5}{3} \pi$.
- (16) Si ha $\iint_D \sin(x+y) \, dx dy = \int_0^{\pi/2} \left(\int_0^x \sin(x+y) dy\right) dx = \int_0^{\pi/2} [-\cos(x+y)]_{y=0}^{y=x} dx = \int_0^{\pi/2} (\cos x - \cos(2x)) dx = \left[\sin x - \frac{1}{2} \sin(2x)\right]_0^{\pi/2} = 1$.

(17) Si ha $\iint_D y \cos x \, dx dy = \int_{-\sqrt{\pi}}^{2\sqrt{\pi}} y [\sin x]_{x=0}^{x=y^2+\pi} dy = -\int_{-\sqrt{\pi}}^{2\sqrt{\pi}} y \sin(y^2) dy \stackrel{(a)}{=} -\frac{1}{2} \int_{-\pi}^{4\pi} \sin t \, dt = \frac{1}{2} [\cos t]_{-\pi}^{4\pi} = 1$, dove in (a) si è usato il cambio di variabile $y^2 = t \implies 2y dy = dt$. □

Svolgimento esercizio 2

(1) Si ha $\iint_D \frac{|x|}{\sqrt{1+y^2}} \, dx dy = \int_{-1}^1 |x| \, dx \cdot \int_0^1 \frac{dy}{\sqrt{1+y^2}} \stackrel{(a)}{=} 2 \int_0^1 x \, dx \cdot \int_1^{3+\sqrt{10}} \frac{dt}{t} = [x^2]_0^1 \cdot [\log t]_1^{3+\sqrt{10}} = \log(3+\sqrt{10})$, dove in (a) si è usato il cambio di variabile $\sqrt{1+y^2} = t - y \implies \sqrt{1+y^2} = \frac{t^2+1}{2t}$, $dy = \frac{t^2+1}{2t^2} dt$.

(2) Si ha $\iint_D \frac{1}{(x-y+6)^2} \, dx dy = \int_0^1 [\frac{1}{x-y+6}]_{y=1}^{y=2} dx = \int_0^1 (\frac{1}{x+4} - \frac{1}{x+5}) dx = [\log \frac{x+4}{x+5}]_0^1 = \log \frac{25}{24}$.

(3) Si ha $\iint_D \frac{1-3x^2}{(x-y+6)^2} \, dx dy = \int_0^1 (1-3x^2) [\frac{1}{x-y+6}]_{y=1}^{y=2} dx = \int_0^1 (1-3x^2) (\frac{1}{x+4} - \frac{1}{x+5}) dx = -\int_0^1 \frac{3x^2-1}{(x+4)(x+5)} dx = -\int_0^1 (3 - \frac{47}{x+4} + \frac{20}{x+5}) dx = [-3x + 47 \log(x+4) - 20 \log(x+5)]_0^1 = -3 - 47 \log 4 + 67 \log 5 - 20 \log 6$.

(4) Si ha $\iint_D \frac{1-3x^2}{x-y+6} \, dx dy = \int_0^1 (1-3x^2) [-\log|x-y+6|]_{y=1}^{y=2} dx = \int_0^1 (3x^2-1) (\log(x+4) - \log(x+5)) dx = [(x^3-x)(\log(x+4) - \log(x+5))]_0^1 - \int_0^1 (x^3-x) (\frac{1}{x+4} - \frac{1}{x+5}) dx = -\int_0^1 \frac{x^3-x}{(x+4)(x+5)} dx = -\int_0^1 (x-9 - \frac{104}{x+4} + \frac{175}{x+5}) dx = [-\frac{1}{2}x^2 + 9x + 104 \log(x+4) - 175 \log(x+5)]_0^1 = \frac{17}{2} - 104 \log 4 + 279 \log 5 - 175 \log 6$.

(5) Si ha $\iint_D \frac{1}{2x+y+1} \, dx dy = \int_0^1 [\log(2x+y+1)]_{y=0}^{y=1} dx = \int_0^1 (\log(2x+2) - \log(2x+1)) dx \stackrel{(a)}{=} \frac{1}{2} \int_4^6 \log t \, dt - \frac{1}{2} \int_3^5 \log t \, dt = \frac{1}{2} [t \log t - t]_4^6 - \frac{1}{2} [t \log t - t]_3^5 = 3 \log 6 - 2 \log 4 - \frac{5}{2} \log 5 + \frac{3}{2} \log 3$, dove in (a) si è eseguito il cambio di variabile $2x+2 = t$, risp. $2x+1 = t$, negli integrali.

(6) Si ha $\iint_D \frac{x}{2x+y+1} \, dx dy = \int_0^1 x [\log(2x+y+1)]_{y=0}^{y=1} dx = \int_0^1 x (\log(2x+2) - \log(2x+1)) dx \stackrel{(a)}{=} \frac{1}{4} \int_4^6 (t-2) \log t \, dt - \frac{1}{4} \int_3^5 (t-1) \log t \, dt = \frac{1}{4} [\frac{1}{2}(t-2)^2 \log t - \frac{1}{2} \int \frac{(t-2)^2}{t} dt]_4^6 - \frac{1}{4} [\frac{1}{2}(t-1)^2 \log t - \frac{1}{2} \int \frac{(t-1)^2}{t} dt]_3^5 = 2 \log 6 - \frac{1}{2} \log 4 - \frac{1}{8} [\frac{1}{2}t^2 - 4t + 4 \log t]_4^6 - 2 \log 5 + \frac{1}{8} \log 3 + \frac{1}{8} [\frac{1}{2}t^2 - 2t + \log t]_3^5 = \frac{1}{4} + \frac{3}{2} \log 6 - \frac{15}{8} \log 5 + \frac{3}{8} \log 3$, dove in (a) si è eseguito il cambio di variabile $2x+2 = t$, risp. $2x+1 = t$, negli integrali.

(7) Si ha $\iint_D \frac{1}{y^2+2x+1} \, dx dy = \int_0^1 [\frac{1}{2} \log(y^2+2x+1)]_{x=1}^{x=2} dy = \frac{1}{2} \int_0^1 (\log(y^2+5) - \log(y^2+3)) dx = \frac{1}{2} [y \log(y^2+5) - y \log(y^2+3)]_0^1 - \frac{1}{2} \int_0^1 (\frac{2y^2}{y^2+5} - \frac{2y^2}{y^2+3}) dy = \frac{1}{2} \log \frac{6}{4} + \int_0^1 (1 + \frac{5}{y^2+5} - 1 - \frac{3}{y^2+3}) dy \stackrel{(a)}{=} \frac{1}{2} \log \frac{3}{2} + [\sqrt{5} \operatorname{arctg} \frac{y}{\sqrt{5}} - \sqrt{3} \operatorname{arctg} \frac{y}{\sqrt{3}}]_0^1 = \frac{1}{2} \log \frac{3}{2} + \sqrt{5} \operatorname{arctg} \frac{1}{\sqrt{5}} - \sqrt{3} \operatorname{arctg} \frac{1}{\sqrt{3}}$, dove in (a) si è usato il fatto che $\int \frac{dt}{t^2+c} = \frac{1}{\sqrt{c}} \operatorname{arctg} \frac{t}{\sqrt{c}}$.

(8) Si ha $\iint_D \frac{1}{y^2+2x+1} \, dx dy = \frac{1}{2} \int_1^2 (\int_0^1 (1 - \frac{y^2+1}{y^2+2x+1}) dx) dy = \frac{1}{2} - \frac{1}{4} \int_0^1 (y^2+1) [\log(y^2+2x+1)]_{x=1}^{x=2} dy = \frac{1}{2} - \frac{1}{4} \int_0^1 (y^2+1) (\log(y^2+5) - \log(y^2+3)) dx \stackrel{(a)}{=} \frac{1}{2} - \frac{1}{4} [(y^3+3y) \log(y^2+5) - \frac{2}{3}(y^3-5y) - 2y - \frac{20}{3\sqrt{5}} \operatorname{arctg} \frac{y}{\sqrt{5}}]_0^1 - \frac{1}{4} [(y^3+3y) \log(y^2+3) - \frac{2}{3}(y^3-3y) - 2y]_0^1 = \frac{1}{2} - \frac{1}{4} [(y^3+3y) \log \frac{y^2+5}{y^2+3} + \frac{4}{3}y - \frac{20}{3\sqrt{5}} \operatorname{arctg} \frac{y}{\sqrt{5}}]_0^1 = \frac{1}{2} - \frac{1}{3} \log \frac{3}{2} - \frac{1}{3} + \frac{5}{3\sqrt{5}} \operatorname{arctg} \frac{1}{\sqrt{5}} = \frac{1}{6} - \frac{1}{3} \log \frac{3}{2} + \frac{\sqrt{5}}{3} \operatorname{arctg} \frac{1}{\sqrt{5}}$, dove in (a) si è usato il fatto che $\int (y^2+1) \log(y^2+c) dy = (\frac{y^3}{3}+y) \log(y^2+c) - \int (\frac{y^3}{3}+y) \frac{2y}{y^2+c} dy = (\frac{y^3}{3}+y) \log(y^2+c) - \int (\frac{2}{3} \frac{y^4-c^2}{y^2+c} + 2 + \frac{2}{3} \frac{c^2-2c}{y^2+c}) dy = (\frac{y^3}{3}+y) \log(y^2+c) - \frac{2}{3}(\frac{y^3}{3}-cy) - 2y - (\frac{2}{3}c^2-2c) \frac{1}{\sqrt{c}} \operatorname{arctg} \frac{y}{\sqrt{c}}$.

- (9) Si ha $\iint_D \log(x+y+5) dx dy = \int_{-2}^1 (\int_x^1 \log(x+y+5) dy) dx = \int_{-2}^1 [(x+y+5) \log(x+y+5) - y]_{y=x}^{y=1} dx = \int_{-2}^1 ((x+6) \log(x+6) - 1 - (2x+5) \log(2x+5) + x) dx \stackrel{(a)}{=} [\frac{1}{2} x^2 - x]_{-2}^1 + \int_4^7 t \log t dt - \frac{1}{2} \int_1^7 t \log t dt \stackrel{(b)}{=} -\frac{9}{2} + [\frac{1}{2} t^2 \log t - \frac{1}{4} t^2]_4^7 - \frac{1}{2} [\frac{1}{2} t^2 \log t - \frac{1}{4} t^2]_1^7 = -\frac{27}{4} + \frac{49}{4} \log 7 - 8 \log 4$, dove si sono usati in (a) i cambi di variabile $x+6=t$, $2x+5=t$, e in (b) il risultato $\int t \log t dt = \frac{1}{2} t^2 \log t - \frac{1}{4} t^2$.
- (10) Si ha $\iint_D \frac{y^2+1}{y^2+2x+1} dx dy = 2 \int_0^1 (\int_y^1 \frac{y^2+1}{y^2+2x+1} dx) dy = \int_0^1 (y^2+1) [\log(y^2+2x+1)]_{x=y}^{x=1} dy = \int_0^1 (y^2+1) (\log(y^2+3) - 2 \log(y+1)) dy = [(\frac{y^3}{3} + y) (\log(y^2+3) - 2 \log(y+1))]_0^1 - \int_0^1 (\frac{y^3}{3} + y) (\frac{2y}{y^2+3} - \frac{2}{y+1}) dy = -\frac{2}{3} \int_0^1 (y^2 - \frac{y(y^2+3)}{y+1}) dy = -\frac{2}{3} \int_0^1 (y - 4 + \frac{4}{y+1}) dy = -\frac{2}{3} [\frac{1}{2} y^2 - 4y + 4 \log(y+1)]_0^1 = \frac{7}{3} - \frac{8}{3} \log 2$.
- (11) Si ha $\iint_D \frac{1}{(1+x+y)^2} dx dy = \int_0^1 (\int_0^{2\sqrt{x}} \frac{1}{(1+x+y)^2} dy) dx = \int_0^1 [-\frac{1}{1+x+y}]_0^{2\sqrt{x}} dx = \int_0^1 (\frac{1}{1+x} - \frac{1}{1+x+2\sqrt{x}}) dx \stackrel{(a)}{=} [\log(1+x)]_0^1 - \int_0^1 \frac{2t dt}{1+2t+t^2} = \log 2 - \int_0^1 (\frac{2}{t+1} - \frac{2}{(t+1)^2}) dt = \log 2 - 2 [\log(t+1) + \frac{1}{t+1}]_0^1 = 1 - \log 2$, dove in (a) si è usato il cambio di variabile $x=t^2$.
- (12) Si ha $\iint_D x(y + \sin(\pi y)) dx dy = \int_0^1 x (\int_0^{2\sqrt{x}} (y + \sin(\pi y)) dy) dx = \int_0^1 x [\frac{1}{2} y^2 - \frac{1}{\pi} \cos \pi y]_0^{2\sqrt{x}} dx = \int_0^1 (2x^2 + \frac{1}{\pi} x - \frac{1}{\pi} x \cos(2\pi\sqrt{x})) dx \stackrel{(a)}{=} [\frac{2}{3} x^3 + \frac{1}{2\pi} x^2]_0^1 - \frac{1}{\pi} \int_0^1 t^2 \cos(2\pi t) \cdot 2t dt \stackrel{(b)}{=} \frac{2}{3} + \frac{1}{2\pi} - \frac{2}{\pi} [\frac{1}{2\pi} t^3 \sin(2\pi t) + \frac{3}{(2\pi)^2} t^2 \cos(2\pi t) - \frac{6}{(2\pi)^3} t \sin(2\pi t) - \frac{6}{(2\pi)^4} \cos(2\pi t)]_0^1 = \frac{2}{3} + \frac{1}{2\pi} - \frac{3}{2\pi^3}$, dove si sono usati in (a) il cambio di variabile $x=t^2$, e in (b) il risultato $\int t^3 \cos(2\pi t) dt = \frac{1}{2\pi} t^3 \sin(2\pi t) + \frac{3}{(2\pi)^2} t^2 \cos(2\pi t) - \frac{6}{(2\pi)^3} t \sin(2\pi t) - \frac{6}{(2\pi)^4} \cos(2\pi t)$.
- (13) Si ha $\iint_D |x|y dx dy = 2 \int_0^{2\pi} y (\int_0^{1-\cos y} x dx) dy = \int_0^{2\pi} y [x^2]_0^{1-\cos y} dy = \int_0^{2\pi} y(1-\cos y)^2 dy = \int_0^{2\pi} y(\frac{3}{2} - 2 \cos y + \frac{1}{2} \cos 2y) dy \stackrel{(a)}{=} [\frac{3}{4} y^2 - 2y \sin y - 2 \cos y + \frac{1}{4} y \sin(2y) + \frac{1}{8} \cos(2y)]_0^{2\pi} = 3\pi^2$, dove in (a) si è usato il risultato $\int y \cos(cy) dy = \frac{1}{c} y \sin(cy) + \frac{1}{c^2} \cos(cy)$.
- (14) Posto $D' := \{(x, y) \in \mathbb{R}^2 : 1 \leq x+y \leq 2, x \geq 0, y \geq 0\}$, si ha $\iint_D (x-1)|y| dx dy = -\iint_{D'} |y| dx dy = -4 \iint_{D'} y dx dy = 2 \int_0^1 [y^2]_0^{1-x} dx - 2 \int_0^1 [y^2]_0^{2-x} dx = 2 \int_0^1 (1-x)^2 dx - 2 \int_0^1 (2-x)^2 dx = 2[x - x^2 + \frac{1}{3} x^3]_0^1 - 2[4x - 2x^2 + \frac{1}{3} x^3]_0^2 = -\frac{14}{3}$.
- (15) Si ha $\int_0^1 (\int_x^{x^2} \frac{1}{2x+y+1} dy) dx = \int_0^1 [\log(2x+y+1)]_{y=x}^{y=x^2} dx = \int_0^1 (2 \log(x+1) - \log(3x+1)) dx = [2x \log(x+1) - x \log(3x+1)]_0^1 - \int_0^1 (\frac{2x}{x+1} - \frac{3x}{3x+1}) dx = \int_0^1 (-1 + \frac{2}{x+1} - \frac{1}{3x+1}) dx = [-x + 2 \log(x+1) - \frac{1}{3} \log(3x+1)]_0^1 = \frac{4}{3} \log 2 - 1$.
- (16) Si ha $\int_0^1 (\int_x^{x^2} \frac{x}{2x+y+1} dy) dx = \int_0^1 x [\log(2x+y+1)]_{y=x}^{y=x^2} dx = \int_0^1 x(2 \log(x+1) - \log(3x+1)) dx = [x^2 \log(x+1) - \frac{1}{2} x^2 \log(3x+1)]_0^1 - \int_0^1 (\frac{x^2}{x+1} - \frac{3}{2} \frac{x^2}{3x+1}) dx = -\int_0^1 (x-1 + \frac{1}{x+1} - \frac{1}{6} (3x-1) - \frac{1}{6} \frac{1}{3x+1}) dx = -[\frac{1}{4} x^2 - \frac{5}{6} x + \log(x+1) - \frac{1}{18} \log(3x+1)]_0^1 = \frac{7}{12} - \frac{8}{9} \log 2$.
- (17) Si ha $\int_0^1 (\int_x^{\sqrt{x}} \frac{1}{y^2+2x+1} dy) dx = \int_0^1 (\int_y^y \frac{1}{y^2+2x+1} dx) dy = \frac{1}{2} \int_0^1 [\log(y^2+2x+1)]_{x=y^2}^{x=y} dy = \frac{1}{2} \int_0^1 (2 \log(y+1) - \log(2y^2+1)) dy = [y \log(y+1) - \frac{1}{2} y \log(2y^2+1)]_0^1 - \int_0^1 (\frac{y}{y+1} - \frac{2y^2}{2y^2+1}) dy = \log 2 - \frac{1}{2} \log 3 - \int_0^1 (-\frac{1}{y+1} + \frac{1}{2y^2+1}) dy = \log 2 - \frac{1}{2} \log 3 + [\log(y+1) - \frac{1}{\sqrt{2}} \arctg(y\sqrt{2})]_0^1 = 2 \log 2 - \frac{1}{2} \log 3 - \frac{1}{\sqrt{2}} \arctg \sqrt{2}$.
- (18) Si ha $\iint_D \frac{x^2}{(y+1)^2} dx dy = \int_{-1}^3 x^2 (\int_x^{2x+3} \frac{1}{(y+1)^2} dy) dx = \int_{-1}^3 [-\frac{1}{y+1}]_{y=x}^{y=2x+3} dx = \int_{-1}^3 x^2 (\frac{1}{x^2+1} - \frac{1}{2x+4}) dx = \int_{-1}^3 (1 - \frac{1}{x^2+1} - \frac{1}{2}(x-2) - \frac{2}{x+2}) dx = [2x - \frac{1}{4} x^2 - 2 \log(x+2) - \arctg x]_{-1}^3 = 6 - \log 5 - \arctg 3 - \frac{\pi}{4}$.

(19) Si ha $\iint_D \frac{1}{2x+y+2} dx dy = \int_{-1}^3 \left(\int_{x^2}^{2x+3} \frac{1}{2x+y+2} dy \right) dx = \int_{-1}^3 [\log(2x+y+2)]_{y=x^2}^{2x+3} dx = \int_{-1}^3 (\log(4x+5) - \log(x^2+2x+2)) dx = [x \log(4x+5) - x \log(x^2+2x+2)]_{-1}^3 - \int_{-1}^3 \left(\frac{4x}{4x+5} - \frac{x(2x+2)}{x^2+2x+2} \right) dx = \int_{-1}^3 \left(1 + \frac{5}{4x+5} - \frac{2x+2}{x^2+2x+2} - \frac{2}{(x+1)^2+1} \right) dx = \left[x + \frac{5}{4} \log(4x+5) - \log(x^2+2x+2) - 2 \operatorname{arctg}(x+1) \right]_{-1}^3 = 4 + \frac{1}{4} \log 17 - 2 \operatorname{arctg} 4.$

(20) Si ha $\iint_D \frac{1}{(2x-y+8)^2} dx dy = \int_{-1}^3 \left(\int_{x^2}^{2x+3} \frac{1}{(2x-y+8)^2} dy \right) dx = \int_{-1}^3 \left[\frac{1}{2x-y+8} \right]_{y=x^2}^{2x+3} dx = \int_{-1}^3 \left(\frac{1}{5} + \frac{1}{x^2-2x-8} \right) dx = \int_{-1}^3 \left(\frac{1}{5} + \frac{1}{6} \frac{1}{x-4} - \frac{1}{6} \frac{1}{x+2} \right) dx = \left[\frac{1}{5} x + \frac{1}{6} \log \frac{4-x}{x+2} \right]_{-1}^3 = \frac{4}{5} - \frac{1}{3} \log 5.$

(21) Si ha $\iint_D \frac{1-3x^2}{(2x-y+8)^2} dx dy = \int_{-1}^3 (1-3x^2) \left(\int_{x^2}^{2x+3} \frac{1}{(2x-y+8)^2} dy \right) dx = \int_{-1}^3 (1-3x^2) \left[\frac{1}{2x-y+8} \right]_{y=x^2}^{2x+3} dx = \int_{-1}^3 (1-3x^2) \left(\frac{1}{5} + \frac{1}{x^2-2x-8} \right) dx = \frac{1}{5} [x - x^3]_{-1}^3 - \int_{-1}^3 \left(3 + \frac{47}{6} \frac{1}{x-4} - \frac{11}{6} \frac{1}{x+2} \right) dx = -\frac{24}{5} - \left[3x + \frac{47}{6} \log(4-x) - \frac{11}{6} \log(x+2) \right]_{-1}^3 = -\frac{84}{5} + \frac{29}{3} \log 5.$

(22) Si ha $\iint_D \frac{1-3x^2}{2x-y+8} dx dy = \int_{-1}^3 (1-3x^2) \left(\int_{x^2}^{2x+3} \frac{1}{2x-y+8} dy \right) dx = \int_{-1}^3 (3x^2-1) [\log|2x-y+8|]_{y=x^2}^{2x+3} dx = \int_{-1}^3 (3x^2-1) (\log 5 - \log|x^2-2x-8|) dx = [(x^3-x)(\log 5 - \log|x^2-2x-8|)]_{-1}^3 + \int_{-1}^3 \frac{(x^3-x)(2x-2)}{x^2-2x-8} dx = 2 \int_{-1}^3 \left(x^2+x+9 + \frac{30}{x-4} - \frac{3}{x+2} \right) dx = 2 \left[\frac{1}{3} x^3 + \frac{1}{2} x^2 + 9x + 30 \log(4-x) - 3 \log(x+2) \right]_{-1}^3 = \frac{296}{3} - 66 \log 5.$

□

Svolgimento esercizio 3

(1) Si ha $\iint_D \sqrt{x^2+y^2} dx dy \stackrel{(a)}{=} \int_0^{2\pi} \left(\int_0^3 \rho \rho d\rho \right) d\vartheta = 2\pi \left[\frac{1}{3} \rho^3 \right]_0^3 = 18\pi$, dove in (a) si è usato il cambio di variabile $x = \rho \cos \vartheta$, $y = \rho \sin \vartheta$.

(2) Si ha $\iint_D 2xy dx dy \stackrel{(a)}{=} \int_{\pi/4}^{7\pi/4} \left(\int_0^3 2\rho^2 \cos \vartheta \sin \vartheta \rho d\rho \right) d\vartheta = [\sin^2 \vartheta]_{\pi/4}^{7\pi/4} \cdot \left[\frac{1}{4} \rho^4 \right]_0^3 = -\frac{81}{8}$, dove in (a) si è usato il cambio di variabile $x = \rho \cos \vartheta$, $y = \rho \sin \vartheta$.

(3) Si ha $\iint_D \left| \frac{x}{y} \right| dx dy \stackrel{(a)}{=} 2 \int_{\pi/4}^{\pi/2} \left(\int_1^4 \frac{\cos \vartheta}{\sin \vartheta} \rho d\rho \right) d\vartheta = [\log|\sin \vartheta|]_{\pi/4}^{\pi/2} \cdot [\rho^2]_1^4 = \frac{15}{2} \log 2$, dove in (a) si sono usate le simmetrie del dominio e della funzione, e il cambio di variabile $x = \rho \cos \vartheta$, $y = \rho \sin \vartheta$.

(4) Si ha $\iint_D \frac{1}{x} dx dy \stackrel{(a)}{=} 2 \int_0^{\pi/4} \left(\int_1^2 \frac{1}{\rho \cos \vartheta} \rho d\rho \right) d\vartheta = 2 \int_0^{\pi/4} \frac{\cos \vartheta}{1-\sin^2 \vartheta} d\vartheta \cdot [\rho]_1^2 \stackrel{(b)}{=} 2 \int_0^{\sqrt{2}/2} \frac{dt}{1-t^2} = 2 \int_0^{\sqrt{2}/2} \left(\frac{1}{2} \frac{1}{t+1} - \frac{1}{2} \frac{1}{t-1} \right) dt = [\log \frac{1+t}{1-t}]_{0}^{\sqrt{2}/2} = 2 \log(2 + \sqrt{2}) - \log 2$, dove in (a) si sono usate le simmetrie del dominio e della funzione, e il cambio di variabile $x = \rho \cos \vartheta$, $y = \rho \sin \vartheta$, e in (b) il cambio di variabile $\sin \vartheta = t \implies \cos \vartheta d\vartheta = dt$.

(5) Si ha $\iint_D (x-y)^2 dx dy \stackrel{(a)}{=} \int_{-2}^2 \left(\int_{-2}^2 (x-y)^2 dy \right) dx - \int_0^{2\pi} \left(\int_0^3 \rho^2 (\cos \vartheta - \sin \vartheta)^2 \rho d\rho \right) d\vartheta = \frac{1}{3} \int_{-2}^2 [(y-x)^3]_{y=-2}^{y=2} dx - \int_0^{2\pi} (1 - 2 \cos \vartheta \sin \vartheta) d\vartheta \cdot \left[\frac{1}{4} \rho^4 \right]_0^3 = \frac{16}{3} \int_{-2}^2 (x+1) dx - \frac{1}{4} [\vartheta - \sin^2 \vartheta]_0^{2\pi} = \frac{64}{3} - \frac{\pi}{2}$, dove in (a) si è usato il cambio di variabile $x = \rho \cos \vartheta$, $y = \rho \sin \vartheta$.

(6) Si ha $\int_0^4 \left(\int_{-\sqrt{4x-x^2}}^{\sqrt{4x-x^2}} \sqrt{y^2+x^2} dy \right) dx \stackrel{(a)}{=} \int_{-\pi/2}^{\pi/2} \left(\int_0^4 \cos \vartheta \rho \rho d\rho \right) d\vartheta = \int_{-\pi/2}^{\pi/2} \left[\frac{1}{3} \rho^3 \right]_0^{4 \cos \vartheta} d\vartheta = \frac{64}{3} \int_{-\pi/2}^{\pi/2} \cos^3 \vartheta d\vartheta = \frac{64}{3} \int_{-\pi/2}^{\pi/2} (1 - \sin^2 \vartheta) \cos \vartheta d\vartheta = \frac{64}{3} \left[\sin \vartheta - \frac{1}{3} \sin^3 \vartheta \right]_{-\pi/2}^{\pi/2} = \frac{256}{9}$, dove in (a) si è usato il cambio di variabile $x = \rho \cos \vartheta$, $y = \rho \sin \vartheta$.

(7) Si ha $\iint_D \frac{x^2}{x^2+y^2} dx dy \stackrel{(a)}{=} \int_0^{2\pi} \left(\int_1^3 \cos^2 \vartheta \rho d\rho \right) d\vartheta = \int_0^{2\pi} \frac{1+\cos 2\vartheta}{2} d\vartheta \cdot \left[\frac{1}{2} \rho^2 \right]_1^3 = 2 \left[\vartheta + \frac{1}{2} \sin 2\vartheta \right]_0^{2\pi} = 4\pi$, dove in (a) si è usato il cambio di variabile $x = \rho \cos \vartheta$, $y = \rho \sin \vartheta$.

(8) Si ha $\iint_D \frac{x^2}{x^2+y^2} dx dy \stackrel{(a)}{=} \iint_{D_{\rho,\vartheta}} \cos^2 \vartheta \rho d\rho d\vartheta = \int_{\pi/6}^{5\pi/6} \cos^2 \vartheta \left(\int_2^{4 \sin \vartheta} \rho d\rho \right) d\vartheta = \int_{\pi/6}^{5\pi/6} \cos^2 \vartheta \left[\frac{1}{2} \rho^2 \right]_2^{4 \sin \vartheta} d\vartheta = \frac{1}{2} \int_{\pi/6}^{5\pi/6} \cos^2 \vartheta (16 \sin^2 \vartheta - 4) d\vartheta = 2 \int_{\pi/6}^{5\pi/6} (\sin^2 2\vartheta - \cos^2 \vartheta) d\vartheta = \int_{\pi/6}^{5\pi/6} (1 - \cos 4\vartheta - 1 - \cos 2\vartheta) d\vartheta = -\left[\frac{1}{4} \sin 4\vartheta + \frac{1}{2} \sin 2\vartheta \right]_{\pi/6}^{5\pi/6} = \frac{3\sqrt{3}}{4}$, dove in (a) si è usato il cambio di variabile $x = \rho \cos \vartheta$, $y = \rho \sin \vartheta$, per cui $D_{\rho\vartheta} = \{(\rho, \vartheta) \in \mathbb{R}^2 : 2 \leq \rho \leq 4 \sin \vartheta, \vartheta \in [\frac{\pi}{6}, \frac{5\pi}{6}]\}$.

(9) Si ha $\iint_D |xy| dx dy \stackrel{(a)}{=} \int_0^{2\pi} \left(\int_1^3 \rho^2 |\cos \vartheta \sin \vartheta| \rho d\rho \right) d\vartheta = \frac{1}{2} \int_0^{2\pi} |\sin 2\vartheta| d\vartheta \cdot \left[\frac{1}{4} \rho^4 \right]_1^3 = 40 \int_0^{\pi/2} \sin 2\vartheta d\vartheta = 40 \left[-\frac{1}{2} \cos 2\vartheta \right]_0^{\pi/2} = 40$, dove in (a) si è usato il cambio di variabile $x = \rho \cos \vartheta$, $y = \rho \sin \vartheta$.

(10) Usando il cambio di variabile $x = \rho \cos \vartheta$, $y = \rho \sin \vartheta$, per cui $D \rightsquigarrow D_{\rho\vartheta} = \{(\rho, \vartheta) \in \mathbb{R}^2 : 0 \leq \rho \leq 6 \sin \vartheta, \vartheta \in [0, \pi]\}$, si ha $\iint_D |xy| dx dy = \iint_{D_{\rho,\vartheta}} \rho^2 |\cos \vartheta \sin \vartheta| \rho d\rho d\vartheta = \int_0^\pi |\cos \vartheta \sin \vartheta| \left(\int_0^{6 \sin \vartheta} \rho^3 d\rho \right) d\vartheta = \int_0^\pi |\cos \vartheta \sin \vartheta| \left[\frac{1}{4} \rho^4 \right]_0^{6 \sin \vartheta} d\vartheta = 324 \int_0^\pi |\cos \vartheta \sin \vartheta| \sin^4 \vartheta d\vartheta = 648 \int_0^{\pi/2} \sin^5 \vartheta \cos \vartheta d\vartheta = 108 \left[\sin^6 \vartheta \right]_0^{\pi/2} = 108$.

(11) Usando il cambio di variabile $x = \rho \cos \vartheta$, $y = \rho \sin \vartheta$, per cui $D \rightsquigarrow D_{\rho\vartheta} = \{(\rho, \vartheta) \in \mathbb{R}^2 : 2 \leq \rho \leq 4 \sin \vartheta, \vartheta \in [\frac{\pi}{6}, \frac{5\pi}{6}]\}$, si ha $\iint_D |xy| dx dy = \iint_{D_{\rho,\vartheta}} \rho^2 |\cos \vartheta \sin \vartheta| \rho d\rho d\vartheta = \int_{\pi/6}^{5\pi/6} |\cos \vartheta \sin \vartheta| \left(\int_2^{4 \sin \vartheta} \rho^3 d\rho \right) d\vartheta = \int_{\pi/6}^{5\pi/6} |\cos \vartheta \sin \vartheta| \left[\frac{1}{4} \rho^4 \right]_2^{4 \sin \vartheta} d\vartheta = \int_{\pi/6}^{5\pi/6} |\cos \vartheta \sin \vartheta| (16 \sin^4 \vartheta - 4) d\vartheta = 8 \int_{\pi/6}^{5\pi/6} (4 \sin^5 \vartheta - \sin \vartheta) \cos \vartheta d\vartheta = 8 \left[\frac{2}{3} \sin^6 \vartheta - \frac{1}{2} \sin^2 \vartheta \right]_{\pi/6}^{\pi/2} = 18$.

(12) Si ha $\iint_D \frac{|xy|e^{x^2+y^2}}{x^2+y^2} dx dy \stackrel{(a)}{=} \int_0^{2\pi} \left(\int_1^3 e^{\rho^2} |\cos \vartheta \sin \vartheta| \rho d\rho \right) d\vartheta = \frac{1}{2} \int_0^{2\pi} |\sin 2\vartheta| d\vartheta \cdot \left[\frac{1}{2} e^{\rho^2} \right]_1^3 = e(e^8 - 1) \int_0^{\pi/2} \sin 2\vartheta d\vartheta = e(e^8 - 1) \left[-\frac{1}{2} \cos 2\vartheta \right]_0^{\pi/2} = \frac{e}{2}(e^8 - 1)$, dove in (a) si è usato il cambio di variabile $x = \rho \cos \vartheta$, $y = \rho \sin \vartheta$.

(13) Si ha $\iint_D \frac{|xy|e^{x^2+y^2}}{x^2+y^2} dx dy \stackrel{(a)}{=} \iint_{D_{\rho,\vartheta}} e^{\rho^2} |\cos \vartheta \sin \vartheta| \rho d\rho d\vartheta = \int_{\pi/6}^{5\pi/6} |\cos \vartheta \sin \vartheta| \left(\int_2^{4 \sin \vartheta} e^{\rho^2} \rho d\rho \right) d\vartheta = \int_{\pi/6}^{5\pi/6} |\cos \vartheta \sin \vartheta| \left[\frac{1}{2} e^{\rho^2} \right]_2^{4 \sin \vartheta} d\vartheta = \frac{1}{2} \int_{\pi/6}^{5\pi/6} |\cos \vartheta \sin \vartheta| (e^{16 \sin^2 \vartheta} - e^4) d\vartheta = \int_{\pi/6}^{\pi/2} (e^{16 \sin^2 \vartheta} - e^4) \sin \vartheta \cos \vartheta d\vartheta \stackrel{(b)}{=} \frac{1}{32} \int_4^{16} e^t dt - e^4 \left[\frac{1}{2} \sin^2 \vartheta \right]_{\pi/6}^{\pi/2} = \frac{1}{32} (e^{16} - e^4) - \frac{3}{8} e^4 = \frac{1}{32} e^{16} - \frac{13}{32} e^4$, dove si è usato in (a) il cambio di variabile $x = \rho \cos \vartheta$, $y = \rho \sin \vartheta$, per cui $D_{\rho\vartheta} = \{(\rho, \vartheta) \in \mathbb{R}^2 : 2 \leq \rho \leq 4 \sin \vartheta, \vartheta \in [\frac{\pi}{6}, \frac{5\pi}{6}]\}$, e in (b) il cambio di variabile $t = 16 \sin^2 \vartheta \implies dt = 32 \sin \vartheta \cos \vartheta d\vartheta$.

(14) Si ha $\iint_D x^3 y dx dy \stackrel{(a)}{=} \int_{\pi/4}^{5\pi/4} \left(\int_0^1 2\sqrt{2} \rho^3 \cos^3 \vartheta \cdot 2\sqrt{2} \rho \sin \vartheta d\rho \right) d\vartheta = 32 \left[-\frac{1}{4} \cos^4 \vartheta \right]_{\pi/4}^{5\pi/4} \cdot \left[\frac{1}{6} \rho^6 \right]_0^1 = 0$, dove in (a) si è usato il cambio di variabile $x = \sqrt{2} \rho \cos \vartheta$, $y = 2\sqrt{2} \rho \sin \vartheta$.

(15) Si ha $\iint_D |y| dx dy \stackrel{(a)}{=} \int_{\pi/4}^{5\pi/4} \left(\int_0^1 2\sqrt{2} \rho |\sin \vartheta| d\rho \right) d\vartheta = 8\sqrt{2} \left(\int_{\pi/4}^\pi \sin \vartheta d\vartheta - \int_\pi^{5\pi/4} \sin \vartheta d\vartheta \right) \cdot \left[\frac{1}{3} \rho^3 \right]_0^1 = \frac{8\sqrt{2}}{3} \left([-\cos \vartheta]_{\pi/4}^\pi + [\cos \vartheta]_\pi^{5\pi/4} \right) = \frac{16\sqrt{2}}{3}$, dove in (a) si è usato il cambio di variabile $x = \sqrt{2} \rho \cos \vartheta$, $y = 2\sqrt{2} \rho \sin \vartheta$.

(16) Usando il cambio di variabili $\begin{cases} u = x + y \\ v = x - y \end{cases} \iff \begin{cases} x = \frac{u+v}{2} \\ y = \frac{u-v}{2} \end{cases}$ si ha $D_{uv} = \{(u, v) \in \mathbb{R}^2 : u - 4 \leq v \leq 0, 0 \leq u \leq 4\}$, e $\iint_D (x - y) \sin(x + y) dx dy = \frac{1}{2} \int_{D_{uv}} v \sin u du dv = \frac{1}{2} \int_0^4 \sin u du \int_{u-4}^0 v dv = \frac{1}{4} \int_0^4 \sin u [v^2]_{u-4}^0 du = -\frac{1}{4} \int_0^4 (u - 4)^2 \sin u du \stackrel{(a)}{=} -\frac{1}{4} \left[-(u - 4)^2 \cos u + 2(u - 4) \sin u + 2 \cos u \right]_0^4 = -\frac{1}{4} (2 \cos 4 + 16 - 2) = -\frac{1}{2} (7 + \cos 4)$, dove in (a) si è usato il risultato $\int (u - 4)^2 \sin u du = -(u - 4)^2 \cos u + \int 2(u - 4) \cos u du = -(u - 4)^2 \cos u + 2(u - 4) \sin u - \int 2 \sin u du = -(u - 4)^2 \cos u + 2(u - 4) \sin u + 2 \cos u$.

(17) Usando il cambio di variabili $\begin{cases} u = x + y \\ v = x - y \end{cases} \iff \begin{cases} x = \frac{u+v}{2} \\ y = \frac{u-v}{2} \end{cases}$ si ha $D_{uv} = \{(u, v) \in \mathbb{R}^2 : 0 \leq u \leq 2, 0 \leq v \leq 2\}$, e $\iint_D (x^2 - y^2) \sin(x+y) dx dy = \frac{1}{2} \int_{D_{uv}} vu \sin u du dv = \frac{1}{2} \int_0^2 v dv \int_0^2 u \sin u du = \int_0^2 u \sin u du = [-u \cos u + \sin u]_0^2 = \sin 2 - 2 \cos 2$.

(18) Usando il cambio di variabili $\begin{cases} 3y - x = s \\ y - 2x = t \end{cases} \iff \begin{cases} x = \frac{1}{5}s - \frac{3}{5}t \\ y = \frac{2}{5}s - \frac{1}{5}t \end{cases}$ si ha $D_{st} = \{(s, t) \in \mathbb{R}^2 : s \in [0, 5], t \in [-5, 0]\}$, e $\iint_D \sqrt{x+y} dx dy = \frac{1}{5\sqrt{5}} \int_{D_{st}} \sqrt{3s-4t} ds dt = \frac{1}{5\sqrt{5}} \int_{-5}^0 \left[\frac{2}{9}(3s-4t)^{3/2} \right]_{s=0}^{s=5} dt = \frac{2}{45\sqrt{5}} \int_{-5}^0 ((15-4t)^{3/2} - (-4t)^{3/2}) dt = \frac{2}{45\sqrt{5}} \left[-\frac{1}{10}(15-4t)^{5/2} + \frac{1}{10}(-4t)^{5/2} \right]_{-5}^0 = \frac{49\sqrt{7}}{9} - \sqrt{3} - \frac{32}{9}$.

□

Svolgimento esercizio 4

(1) Si ha area $D = \int_{-1}^1 \left(\int_{x^2}^1 dy \right) dx = \int_{-1}^1 (1 - x^2) dx = \left[x - \frac{1}{3}x^3 \right]_{-1}^1 = \frac{4}{3}$.

(2) Si ha area $D = \int_{-1/2}^2 \left(\int_{x^2}^{x+2} dy \right) dx = \int_{-1/2}^2 (x+2-x^2) dx = \left[2x + \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_{-1/2}^2 = \frac{13}{3}$.

(3) Si ha area $D = 1 + \int_{1/2}^2 \left(\int_0^{1/x} dy \right) dx = 1 + \int_{1/2}^2 \frac{1}{x} dx = 1 + \left[\log|x| \right]_{1/2}^2 = 1 + 2 \log 2$.

(4) Si ha area $D = \int_0^8 \left(\int_{y^2/4}^{2y} dx \right) dy = \int_0^8 \left(2y - \frac{1}{4}y^2 \right) dy = \left[y^2 - \frac{1}{12}y^3 \right]_0^8 = \frac{64}{3}$.

(5) Poiché $D = \{(x, y) \in \mathbb{R}^2 : (x^2 + y^2)^3 \leq 16x^2\} = \{(x, y) \in \mathbb{R}^2 : |y| \leq \sqrt{2^{4/3}x^{2/3} - x^2}\}$, si ha area $D = 4 \int_0^2 \left(\int_0^{\sqrt{2^{4/3}x^{2/3} - x^2}} dy \right) dx = 4 \int_0^2 \sqrt{2^{4/3}x^{2/3} - x^2} dx = 4 \int_0^2 x^{1/3} \sqrt{2^{4/3} - x^{4/3}} dx \stackrel{(a)}{=} 3 \int_0^{2^{4/3}} \sqrt{t} dt = \left[\frac{2}{3}t^{3/2} \right]_0^{2^{4/3}} = 8$, dove in (a) si è usato il cambio di variabile $t = 2^{4/3} - x^{4/3} \implies dt = -\frac{4}{3}x^{1/3}$.

(6) Usando il cambio di coordinate $x = \varrho \cos \vartheta, y = \varrho \sin \vartheta$, per cui $D \rightsquigarrow D_{\varrho\vartheta} = \{(\varrho, \vartheta) \in \mathbb{R}^2 : 3 \leq \varrho \leq 8 \sin \vartheta, \vartheta \in [\arcsin \frac{3}{8}, \pi - \arcsin \frac{3}{8}]\}$, si ha area $D = \int_{\arcsin 3/8}^{\pi - \arcsin 3/8} \left(\int_3^{8 \sin \vartheta} \varrho d\varrho \right) d\vartheta = \int_{\arcsin 3/8}^{\pi - \arcsin 3/8} \left[\frac{1}{2} \varrho^2 \right]_3^{8 \sin \vartheta} d\vartheta = \frac{1}{2} \int_{\arcsin 3/8}^{\pi - \arcsin 3/8} (64 \sin^2 \vartheta - 9) d\vartheta = \frac{1}{2} \int_{\arcsin 3/8}^{\pi - \arcsin 3/8} (23 - 32 \cos 2\vartheta) d\vartheta = \frac{1}{2} [23\vartheta - 16 \sin 2\vartheta]_{\arcsin 3/8}^{\pi - \arcsin 3/8} = \frac{23}{2} \pi - 23 \arcsin \frac{3}{8} - \frac{3}{2} \sqrt{55}$.

(7) Usando il cambio di coordinate $x = \sqrt{2}\varrho \cos \vartheta, y = 2\sqrt{2}\varrho \sin \vartheta$, per cui $D \rightsquigarrow D_{\varrho\vartheta} = \{(\varrho, \vartheta) \in \mathbb{R}^2 : \varrho \in [0, 1], \vartheta \in [\frac{\pi}{4}, \frac{5\pi}{4}]\}$, si ha area $D = \int_{D_{\varrho\vartheta}} 4\varrho d\varrho d\vartheta = \int_{\pi/4}^{5\pi/4} \left(\int_0^1 4\varrho d\varrho \right) d\vartheta = \pi [2\varrho^2]_0^1 = 2\pi$.

□

Svolgimento esercizio 5

(1) Si ha $\iiint_D \frac{x^2 y}{z} dx dy dz = \int_0^1 x^2 dx \cdot \int_0^1 y dy \cdot \int_1^2 \frac{1}{z} dz = \left[\frac{1}{3}x^3 \right]_0^1 \cdot \left[\frac{1}{2}y^2 \right]_0^1 \cdot [\log z]_1^2 = \frac{1}{6} \log 2$.

(2) Si ha $\iiint_D x \sin^2 y \cos z dx dy dz = \int_0^1 x dx \cdot \int_0^\pi \sin^2 y dy \cdot \int_0^{\pi/2} \cos z dz = \left[\frac{1}{2}x^2 \right]_0^1 \cdot \left[\frac{1}{2}y - \frac{1}{4} \sin 2y \right]_0^\pi \cdot [\sin z]_0^{\pi/2} = \frac{\pi}{4}$.

(3) Si ha $\iiint_D 2(x+y+z) dx dy dz = 2 \int_0^1 x dx + 2 \int_0^1 y dy + 2 \int_0^1 z dz = 6 \left[\frac{1}{2}x^2 \right]_0^1 = 3$.

- (4) Posto $A_z = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq z^2\}$, si ha $\iiint_D (x + y + z) dx dy dz = \iiint_D z dx dy dz = \int_{-1}^2 z (\int_{A_z} dx dy) dz = \pi \int_{-1}^2 z^3 dz = \pi [\frac{1}{4} z^4]_{-1}^2 = \frac{15}{4} \pi$.
- (5) Si ha $\iiint_D (x^3 + y^3 + z^3) dx dy dz = 3 \int_0^1 x^3 (\int_0^{1-x} (\int_0^{1-x-y} dz) dy) dx = 3 \int_0^1 x^3 (\int_0^{1-x} (1-x-y) dy) dx = 3 \int_0^1 x^3 [(1-x)y - \frac{1}{2} y^2]_0^{1-x} dx = \frac{3}{2} \int_0^1 x^3 (1-x)^2 dx = \frac{3}{2} [\frac{1}{6} x^6 - \frac{2}{5} x^5 + \frac{1}{4} x^4]_0^1 = \frac{1}{40}$.
- (6) Si ha $\iiint_D \frac{1}{\sqrt{1-z^2}} dx dy dz = \int_0^{1/2} \frac{1}{\sqrt{1-z^2}} (\int_0^z (\int_0^y dx) dy) dz = \int_0^{1/2} \frac{1}{\sqrt{1-z^2}} (\int_0^z y dy) dz = \frac{1}{2} \int_0^{1/2} \frac{1}{\sqrt{1-z^2}} [y^2]_0^z dz = \frac{1}{2} \int_0^{1/2} \frac{z^2}{\sqrt{1-z^2}} dz \stackrel{(a)}{=} \frac{1}{2} \int_0^{\pi/6} \sin^2 t dt = \frac{1}{2} [\frac{1}{2} t - \frac{1}{4} \sin 2t]_0^{\pi/6} = \frac{\pi}{24} - \frac{\sqrt{3}}{16}$, dove in (a) si è usato il cambio di variabili $z = \sin t \implies dz = \cos t dt$.
- (7) Si ha $\iiint_D xy z^2 dx dy dz = \int_0^1 x (\int_{-x}^x z^2 (\int_{x+z}^4 y dy) dz) dx = \frac{1}{2} \int_0^1 x (\int_{-x}^x z^2 (4 - (x+z)^2) dz) dx = \frac{1}{2} \int_0^1 x (\int_{-x}^x z^2 ((4-x^2) - 2xz - z^2) dz) dx = \int_0^1 x [\frac{1}{3} (4-x^2) z^3 - \frac{1}{5} z^5]_0^x dx = \int_0^1 (\frac{1}{3} (4-x^2) x^4 - \frac{1}{5} x^6) dx = [\frac{4}{3 \cdot 5} x^5 - \frac{8}{15 \cdot 7} x^7]_0^1 = \frac{4}{21}$.
- (8) Si ha $\iiint_D (x^2 + y^2 + z^2) dx dy dz = \int_0^1 (\int_0^{1-x} (\int_0^{2(1-x-y)} (x^2 + y^2 + z^2) dz) dy) dx = \int_0^1 (\int_0^{1-x} [x^2 z + y^2 z + \frac{1}{3} z^3]_{z=0}^{z=2(1-x-y)} dy) dx = \int_0^1 (\int_0^{1-x} (2(x^2 + y^2)(1-x-y) + \frac{8}{3} (1-x-y)^3) dy) dx = \int_0^1 (\int_0^{1-x} (2x^2(1-x) - 2x^2 y + 2(1-x)y^2 - 2y^3 + \frac{8}{3} (1-x-y)^3) dy) dx = \int_0^1 [2x^2(1-x)y - x^2 y^2 + \frac{2}{3} (1-x)y^3 - \frac{1}{2} y^4 - \frac{8}{3} \cdot \frac{1}{4} (1-x-y)^4]_{y=0}^{y=1-x} dx = \int_0^1 (2x^2(1-x)^2 - x^2(1-x)^2 + \frac{2}{3} (1-x)^4 - \frac{1}{2} (1-x)^4 + \frac{2}{3} (1-x)^4) dx = \int_0^1 (x^2(1-x)^2 + \frac{5}{6} (1-x)^4) dx = [\frac{1}{3} x^3 - \frac{1}{2} x^4 + \frac{1}{5} x^5 + \frac{1}{6} (x-1)^5]_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} = \frac{1}{5}$.
- (9) Posto $A = \{(x, y) : 0 \leq y \leq x; \frac{1}{16} \leq x^2 + y^2 \leq 1\}$ si ottiene $\iiint_D \frac{x}{\sqrt{x^2 + y^2}} dx dy dz = \int_A (\int_0^1 \frac{x}{\sqrt{x^2 + y^2}} dz) dx dy = \int_A \frac{x}{\sqrt{x^2 + y^2}} dx dy \stackrel{(a)}{=} \int_0^{\pi/4} \int_{1/4}^1 \cos \vartheta \varrho d\varrho d\vartheta = [\sin \vartheta]_0^{\pi/4} [\frac{1}{2} \varrho^2]_{1/4}^1 = \frac{\sqrt{2}}{2} \cdot \frac{1}{2} (1 - \frac{1}{16}) = \frac{15\sqrt{2}}{64}$, dove in (a) si è usato il cambiamento di coordinate polari.
- (10) Posto $A = \{(x, y) : 0 \leq y \leq x; \frac{1}{16} \leq x^2 + y^2 \leq 1\}$ si ottiene $\iiint_D \frac{2z}{(x^2 + y^2)^{3/2}} \cos(\frac{\arctg(y/x)}{\sqrt{x^2 + y^2}}) dx dy dz = \int_A (\int_0^1 \frac{1}{(x^2 + y^2)^{3/2}} \cos(\frac{\arctan(y/x)}{\sqrt{x^2 + y^2}}) 2z dz) dx dy = \int_A \frac{1}{(x^2 + y^2)^{3/2}} \cos(\frac{\arctan(y/x)}{\sqrt{x^2 + y^2}}) dx dy = (\text{coord. polari}) = \int_0^{\pi/4} \int_{1/4}^1 \frac{1}{\rho^3} \cos \frac{\theta}{\sqrt{\rho}} \rho d\rho d\theta = \int_{1/4}^1 (\int_0^{\pi/4} \frac{1}{\rho^2} \cos \frac{\theta}{\sqrt{\rho}} d\theta) d\rho = \int_{1/4}^1 (\frac{1}{\rho^{3/2}} [\sin \frac{\theta}{\sqrt{\rho}}]_{\theta=0}^{\theta=\pi/4}) d\rho = \int_{1/4}^1 \frac{1}{\rho^{3/2}} \sin \frac{\pi}{4\sqrt{\rho}} d\rho \stackrel{(a)}{=} \frac{8}{\pi} [\cos \frac{\pi}{4\sqrt{\rho}}]_{1/4}^1 = \frac{8}{\pi} (\cos \frac{\pi}{4} - \cos \frac{\pi}{2}) = \frac{4\sqrt{2}}{\pi}$, dove in (a) si è usato il cambiamento di coordinate $s = \frac{\pi}{4\sqrt{\rho}} \implies ds = -\frac{\pi}{8\rho^{3/2}} d\rho$.
- (11) Si ha $\iiint_D x^2 dx dy dz \stackrel{(a)}{=} \int_0^\pi \int_0^{2\pi} \int_0^1 \varrho^2 \cos^2 \vartheta \sin^2 \varphi \varrho^2 \sin \varphi d\varrho d\vartheta d\varphi = \int_0^\pi \sin^3 \varphi d\varphi \cdot \int_0^{2\pi} \cos^2 \vartheta d\vartheta \cdot \int_0^1 \varrho^4 d\varrho = \int_0^\pi (1 - \cos^2 \varphi) \sin \varphi d\varphi \cdot \int_0^{2\pi} \frac{1}{2} (1 + \cos 2\vartheta) d\vartheta \cdot [\frac{1}{5} \varrho^5]_0^1 = \frac{1}{5} [\frac{1}{3} \cos^3 \varphi - \cos \varphi]_\pi^0 \cdot [\frac{1}{2} \vartheta + \frac{1}{4} \sin 2\vartheta]_0^{2\pi} = \frac{4\pi}{15}$, dove si è usato in (a) il cambio di variabile $x = \varrho \sin \varphi \cos \vartheta, y = \varrho \sin \varphi \sin \vartheta, z = \varrho \cos \varphi$, per cui $D \rightsquigarrow D_{\varrho\vartheta\varphi} = \{(\varrho, \vartheta, \varphi) \in \mathbb{R}^3 : 0 \leq \varrho \leq 1, 0 \leq \varphi \leq \pi, 0 \leq \vartheta \leq 2\pi\}$.
- (12) Si ha $\iiint_D (y-3)^2 dx dy dz \stackrel{(a)}{=} \int_0^\pi \int_0^{2\pi} \int_0^2 (\varrho \sin \vartheta \sin \varphi - 3)^2 \varrho^2 \sin \varphi d\varrho d\vartheta d\varphi = \int_0^\pi \int_0^{2\pi} \int_0^2 (\varrho^4 \sin^2 \vartheta \sin^3 \varphi - 6\varrho^3 \sin \vartheta \sin^2 \varphi + 9\varrho^2 \sin \varphi) d\varrho d\vartheta d\varphi = \int_0^\pi \sin^3 \varphi d\varphi \cdot \int_0^{2\pi} \sin^2 \vartheta d\vartheta \cdot \int_0^2 \varrho^4 d\varrho + 9 \int_0^\pi \sin \varphi d\varphi \cdot \int_0^{2\pi} d\vartheta \cdot \int_0^2 \varrho^2 d\varrho = \int_0^\pi (1 - \cos^2 \varphi) \sin \varphi d\varphi \cdot \int_0^{2\pi} \frac{1}{2} (1 - \cos 2\vartheta) d\vartheta \cdot [\frac{1}{5} \varrho^5]_0^2 + 18\pi [-\cos \varphi]_0^\pi \cdot [\frac{1}{3} \varrho^3]_0^2 = \frac{32}{5} [\frac{1}{3} \cos^3 \varphi - \cos \varphi]_\pi^0 \cdot [\frac{1}{2} \vartheta + \frac{1}{4} \sin 2\vartheta]_0^{2\pi} + 96\pi = \frac{1568\pi}{15}$, dove si è usato in (a) il cambio di variabile $x = \varrho \sin \varphi \cos \vartheta, y = \varrho \sin \varphi \sin \vartheta, z = \varrho \cos \varphi$, per cui $D \rightsquigarrow D_{\varrho\vartheta\varphi} = \{(\varrho, \vartheta, \varphi) \in \mathbb{R}^3 : 0 \leq \varrho \leq 2, 0 \leq \varphi \leq \pi, 0 \leq \vartheta \leq 2\pi\}$.

(13) Si ha $\iiint_D x^2 y \, dx \, dy \, dz \stackrel{(a)}{=} \int_0^\pi \int_0^\pi \int_0^1 \varrho^2 \cos^2 \vartheta \sin^2 \varphi \cdot \varrho \sin \vartheta \sin \varphi \varrho^2 \sin \varphi \, d\varrho \, d\vartheta \, d\varphi = \int_0^\pi \sin^4 \varphi \, d\varphi \cdot \int_0^\pi \cos^2 \vartheta \sin \vartheta \, d\vartheta \cdot \int_0^1 \varrho^5 \, d\varrho = \int_0^\pi \left(\frac{3}{8} - \frac{1}{2} \cos 2\varphi + \frac{1}{8} \cos 4\varphi \right) d\varphi \cdot \left[-\frac{1}{3} \cos^3 \vartheta \right]_0^\pi \cdot \left[\frac{1}{6} \varrho^6 \right]_0^1 = \frac{1}{9} \left[\frac{3}{8} \varphi - \frac{1}{4} \sin 2\varphi + \frac{1}{32} \sin 4\varphi \right]_0^\pi = \frac{\pi}{24}$, dove si è usato in (a) il cambio di variabile $x = \varrho \sin \varphi \cos \vartheta, y = \varrho \sin \varphi \sin \vartheta, z = \varrho \cos \varphi$, per cui $D \rightsquigarrow D_{\varrho\vartheta\varphi} = \{(\varrho, \vartheta, \varphi) \in \mathbb{R}^3 : 0 \leq \varrho \leq 1, 0 \leq \varphi \leq \pi, 0 \leq \vartheta \leq \pi\}$.

(14) Si ha $\iiint_D \left(e^z + \frac{2y}{1+x^2+y^2} + x - 1 \right) dx \, dy \, dz \stackrel{(a)}{=} \iiint_D e^z \, dx \, dy \, dz \stackrel{(b)}{=} \int_0^{2\pi} \int_0^\pi \int_0^1 e^{\varrho \cos \varphi} \varrho^2 \sin \varphi \, d\varrho \, d\vartheta \, d\varphi = 2\pi \int_0^1 \varrho \left(\int_0^\pi e^{\varrho \cos \varphi} \varrho \sin \varphi \, d\varphi \right) d\varrho \stackrel{(c)}{=} 2\pi \int_0^1 \varrho \left[-e^{\varrho \cos \varphi} \right]_{\varphi=0}^{\varphi=\pi} d\varrho = 2\pi \int_0^1 \varrho (e^\varrho - e^{-\varrho}) d\varrho \stackrel{(d)}{=} 2\pi \left[\varrho(e^\varrho + e^{-\varrho}) - e^\varrho + e^{-\varrho} \right]_0^1 = \frac{4\pi}{e}$, dove si è usato in (a) il fatto che D è invariante rispetto alle trasformazioni $y \rightarrow -y$ e $x \rightarrow 2 - x$, mentre il secondo termine cambia segno sotto la trasformazione $y \rightarrow -y$, e il terzo cambia segno sotto la trasformazione $x \rightarrow 2 - x$, in (b) il cambio di variabile $x = 1 + \varrho \sin \varphi \cos \vartheta, y = \varrho \sin \varphi \sin \vartheta, z = \varrho \cos \varphi$, per cui $D \rightsquigarrow D_{\varrho\vartheta\varphi} = \{(\varrho, \vartheta, \varphi) \in \mathbb{R}^3 : 0 \leq \varrho \leq 1, 0 \leq \varphi \leq \pi, 0 \leq \vartheta \leq 2\pi\}$, in (c) il cambiamento di variabile $\varrho \cos \varphi = t \implies \varrho \sin \varphi \, d\varphi = -dt$, e in (d) il risultato $\int \varrho (e^\varrho - e^{-\varrho}) d\varrho = \varrho(e^\varrho + e^{-\varrho}) - \int (e^\varrho + e^{-\varrho}) d\varrho = \varrho(e^\varrho + e^{-\varrho}) - e^\varrho + e^{-\varrho}$.

(15) Si ha $\iiint_D \left(e^z + z(x-1)^2 + \frac{2z}{1+(x-1)^2+(y/2)^2+z^2} \right) dx \, dy \, dz \stackrel{(a)}{=} \iiint_D e^z \, dx \, dy \, dz \stackrel{(b)}{=} \int_0^{2\pi} \int_0^\pi \int_0^1 e^{\varrho \cos \varphi} \cdot 2\varrho^2 \sin \varphi \, d\varrho \, d\vartheta \, d\varphi = 4\pi \int_0^1 \varrho \left(\int_0^\pi e^{\varrho \cos \varphi} \varrho \sin \varphi \, d\varphi \right) d\varrho \stackrel{(c)}{=} 4\pi \int_0^1 \varrho \left[-e^{\varrho \cos \varphi} \right]_{\varphi=0}^{\varphi=\pi} d\varrho = 2\pi \int_0^1 \varrho (e^\varrho - e^{-\varrho}) d\varrho \stackrel{(d)}{=} 4\pi \left[\varrho(e^\varrho + e^{-\varrho}) - e^\varrho + e^{-\varrho} \right]_0^1 = \frac{8\pi}{e}$, dove si è usato in (a) il fatto che D è invariante rispetto alla trasformazione $z \rightarrow -z$, mentre il secondo e il terzo termine cambiano segno sotto la medesima trasformazione, in (b) il cambio di coordinate ellittiche $x = 1 + \varrho \sin \varphi \cos \vartheta, y = 2\varrho \sin \varphi \sin \vartheta, z = \varrho \cos \varphi$, per cui $D \rightsquigarrow D_{\varrho\vartheta\varphi} = \{(\varrho, \vartheta, \varphi) \in \mathbb{R}^3 : 0 \leq \varrho \leq 1, 0 \leq \varphi \leq \pi, 0 \leq \vartheta \leq 2\pi\}$, e il determinante jacobiano della trasformazione vale $2\varrho^2 \sin \varphi$, in (c) si è usato il cambiamento di variabile $\varrho \cos \varphi = t \implies \varrho \sin \varphi \, d\varphi = -dt$, e in (d) si è usato il risultato $\int \varrho (e^\varrho - e^{-\varrho}) d\varrho = \varrho(e^\varrho + e^{-\varrho}) - \int (e^\varrho + e^{-\varrho}) d\varrho = \varrho(e^\varrho + e^{-\varrho}) - e^\varrho + e^{-\varrho}$. □

Svolgimento esercizio 6

(1) Posto $A_z := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4 - z^2\}$, si ha $\text{vol } D = \int_1^2 \left(\iint_{A_z} dx \, dy \right) dz = \int_1^2 \pi(4 - z^2) dz = \pi \left[4z - \frac{1}{3} z^3 \right]_1^2 = \frac{5}{3} \pi$.

(2) Posto $A_z := \{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 \leq 1 + 9z^2\}$, si ha $\text{vol } D = \int_{-1}^1 \left(\iint_{A_z} dx \, dy \right) dz = \pi \int_{-1}^1 \sqrt{1 + 9z^2} \cdot \frac{1}{2} \sqrt{1 + 9z^2} dz = \frac{\pi}{2} \int_{-1}^1 (1 + 9z^2) dz = \frac{\pi}{2} \left[z + 3z^3 \right]_{-1}^1 = 4\pi$.

(3) Posto $A := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, si ha $\text{vol } D = \iint_A \left(\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz \right) dx \, dy = 2 \int_A \sqrt{4 - x^2 - y^2} \, dx \, dy = 2 \int_0^{2\pi} \left(\int_0^1 \sqrt{4 - \varrho^2} \varrho \, d\varrho \right) d\vartheta \stackrel{(a)}{=} 4\pi \int_3^4 \sqrt{t} \frac{1}{2} dt = 2\pi \left[\frac{2}{3} t^{3/2} \right]_3^4 = \frac{4\pi}{3} (8 - 3\sqrt{3})$, dove in (a) si è usato il cambio di variabile $t = 4 - \varrho^2 \implies dt = -2\varrho \, d\varrho$.

(4) Posto $A := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, si ha $\text{vol } D = \iint_A \left(\int_0^{x^2+y^2} dz \right) dx \, dy = \int_A (x^2 + y^2) \, dx \, dy = \int_0^{2\pi} \left(\int_0^1 \varrho^2 \varrho \, d\varrho \right) d\vartheta = 2\pi \left[\frac{1}{4} \varrho^4 \right]_0^1 = \frac{\pi}{2}$.

(5) Posto $A := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, si ha $\text{vol } D = \iint_A \left(\int_{-1}^{x^2-y^2} dz \right) dx \, dy = \int_A (x^2 - y^2 + 1) \, dx \, dy = \int_0^{2\pi} \left(\int_0^1 (\varrho^2 \cos^2 \vartheta - \varrho^2 \sin^2 \vartheta + 1) \varrho \, d\varrho \right) d\vartheta = \int_0^{2\pi} \left(\frac{1}{3} \cos^2 \vartheta - \frac{1}{3} \sin^2 \vartheta + 1 \right) d\vartheta = \int_0^{2\pi} \left(\frac{1}{3} \cos 2\vartheta + 1 \right) d\vartheta = \left[\frac{1}{6} \sin 2\vartheta + \vartheta \right]_0^{2\pi} = 2\pi$.

(6) Posto $A := \{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 \leq 1\}$, si ha $\text{vol } D = \iint_A \left(\int_{-\frac{1}{3}\sqrt{1-x^2-4y^2}}^{\frac{1}{3}\sqrt{1-x^2-4y^2}} dz \right) dx dy = \frac{2}{3} \int_A \sqrt{1-x^2-4y^2} dx dy \stackrel{(a)}{=} \frac{2}{3} \int_0^{2\pi} \left(\int_0^1 \sqrt{1-\rho^2} \frac{1}{2} \rho d\rho \right) d\vartheta \stackrel{(b)}{=} \frac{2\pi}{3} \int_0^1 \sqrt{t} \frac{1}{2} dt = \frac{\pi}{3} \left[\frac{2}{3} t^{3/2} \right]_0^1 = \frac{2}{9} \pi$, dove si è usato in (a) il cambio di coordinate $x = \rho \cos \vartheta$, $y = \frac{1}{2} \rho \sin \vartheta$, e in (b) il cambio di coordinate $t = 1 - \rho^2 \implies dt = -2\rho d\rho$.

(7) Posto $D_1 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1, z \geq 0\}$ e $D_2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z+1)^2 \leq 2, z \geq 0\}$, si ha $D = D_1 \setminus D_2$, e $\text{vol } D = \text{vol } D_1 - \text{vol } D_2$. Si ha $\text{vol } D_1 = \frac{2}{3} \pi$. Per calcolare $\text{vol } D_2$, poniamo $A_z = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2 - (z+1)^2\}$, per cui $\text{vol } D_2 = \int_0^{\sqrt{2}-1} \left(\iint_{A_z} dx dy \right) dz = \pi \int_0^{\sqrt{2}-1} (1 - 2z - z^2) dz = \left[z - z^2 - \frac{1}{3} z^3 \right]_0^{\sqrt{2}-1} = \frac{\pi}{3} (4\sqrt{2} - 5)$. Allora $\text{vol } D = \text{vol } D_1 - \text{vol } D_2 = \frac{\pi}{3} (7 - 4\sqrt{2})$. □

Svolgimento esercizio 7

(1) Posto $D^a = \{(x, y) \in \mathbb{R}^2 : y \in [a, 1], -y^2 \leq x \leq y^2\}$, si ha $\iint_D \frac{\arctg y^2}{y} dx dy = \lim_{a \rightarrow 0^+} \iint_{D^a} \frac{\arctg y^2}{y} dx dy = \lim_{a \rightarrow 0^+} \int_a^1 \frac{\arctg y^2}{y} \left(\int_{-y^2}^{y^2} dx \right) dy = 2 \lim_{a \rightarrow 0^+} \int_a^1 y \arctg y^2 dy = \lim_{a \rightarrow 0^+} [\arctg y^2]_a^1 = \lim_{a \rightarrow 0^+} \left(\frac{\pi}{4} - \arctg a^2 \right) = \frac{\pi}{4}$.

(2) Posto $D^a = \{(x, y) \in \mathbb{R}^2 : y \in [a, 1], -y^2 \leq x \leq y^2\}$, si ha $\iint_D \frac{\arctg y}{y^2} dx dy = \lim_{a \rightarrow 0^+} \iint_{D^a} \frac{\arctg y}{y^2} dx dy = \lim_{a \rightarrow 0^+} \int_a^1 \frac{\arctg y}{y^2} \left(\int_{-y^2}^{y^2} dx \right) dy = 2 \lim_{a \rightarrow 0^+} \int_a^1 \arctg y dy = 2 \lim_{a \rightarrow 0^+} \left[y \arctg y - \frac{1}{2} \log(y^2 + 1) \right]_a^1 = 2 \lim_{a \rightarrow 0^+} \left(\frac{\pi}{4} - \frac{1}{2} \log 2 - a \arctg a + \frac{1}{2} \log(a^2 + 1) \right) = \frac{\pi}{2} - \log 2$.

(3) Posto $D^a = \{(x, y) \in \mathbb{R}^2 : a \leq |y| \leq 1, y^2 \leq x \leq 1\}$, si ha $\iint_D \frac{1}{(y^2+x)^2} dx dy = \lim_{a \rightarrow 0^+} \iint_{D^a} \frac{1}{(y^2+x)^2} dx dy = 2 \lim_{a \rightarrow 0^+} \int_a^1 \left(\int_{y^2}^1 \frac{1}{(y^2+x)^2} dx \right) dy = -2 \lim_{a \rightarrow 0^+} \int_a^1 \left[\frac{1}{y^2+x} \right]_{x=y^2}^{x=1} dy = -2 \lim_{a \rightarrow 0^+} \int_a^1 \left(\frac{1}{y^2+1} - \frac{1}{2y^2} \right) dy = -2 \lim_{a \rightarrow 0^+} \left[\arctg y + \frac{1}{2y} \right]_a^1 = -2 \lim_{a \rightarrow 0^+} \left(\frac{\pi}{4} + \frac{1}{2} - \arctg a - \frac{1}{2a} \right) = +\infty$.

(4) Posto $D^a = \{(x, y) \in \mathbb{R}^2 : -\frac{1}{2} + a \leq x \leq 0, |y| \leq \sqrt{2x+1}\}$, si ha $\iint_D \frac{(x+2)|y|}{y^2+2x+1} dx dy = \lim_{a \rightarrow 0^+} \iint_{D^a} \frac{(x+2)|y|}{y^2+2x+1} dx dy = \lim_{a \rightarrow 0^+} \int_{-\frac{1}{2}+a}^0 \left(\int_{-\sqrt{2x+1}}^{\sqrt{2x+1}} \frac{(x+2)|y|}{y^2+2x+1} dy \right) dx = \lim_{a \rightarrow 0^+} \int_{-\frac{1}{2}+a}^0 (x+2) [\log(y^2+2x+1)]_{y=0}^{y=\sqrt{2x+1}} dx = \lim_{a \rightarrow 0^+} \int_{-\frac{1}{2}+a}^0 (x+2) (\log(4x+2) - \log(2x+1)) dx = \log 2 \lim_{a \rightarrow 0^+} \left[\frac{1}{2} x^2 + 2x \right]_{-\frac{1}{2}+a}^0 = -\log 2 \lim_{a \rightarrow 0^+} \left(\frac{1}{2} a^2 + \frac{3}{2} a - \frac{7}{8} \right) dx = \frac{7}{8} \log 2$.

(5) Posto $D^a = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq 1, x \leq y \leq \sqrt{x}\}$, si ha $\iint_D \frac{(x+2)y}{y^2+2x} dx dy = \lim_{a \rightarrow 0^+} \iint_{D^a} \frac{(x+2)y}{y^2+2x} dx dy = \lim_{a \rightarrow 0^+} \int_a^1 \left(\int_x^{\sqrt{x}} \frac{(x+2)y}{y^2+2x} dy \right) dx = \frac{1}{2} \lim_{a \rightarrow 0^+} \int_a^1 (x+2) [\log(y^2+2x)]_{y=x}^{y=\sqrt{x}} dx = \frac{1}{2} \lim_{a \rightarrow 0^+} \int_a^1 (x+2) (\log(3x) - \log(x^2+2x)) dx = \frac{1}{2} \lim_{a \rightarrow 0^+} \left[\left(\frac{1}{2} x^2 + 2x \right) (\log(3x) - \log(x^2+2x)) \right]_a^1 - \frac{1}{2} \lim_{a \rightarrow 0^+} \int_a^1 \left(\frac{1}{2} x^2 + 2x \right) \left(\frac{1}{x} - \frac{2x+2}{x^2+2x} \right) dx = -\frac{1}{2} \lim_{a \rightarrow 0^+} \int_a^1 \left(-\frac{1}{2} x - 1 + \frac{2}{x+2} \right) dx = \frac{1}{2} \lim_{a \rightarrow 0^+} \left[\frac{1}{4} x^2 + x - 2 \log(x+2) \right]_a^1 = \frac{5}{8} - 2 \log \frac{3}{2}$.

(6) Posto $D^a = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, x \leq y \leq \sqrt{2x-x^2}, x^2+y^2 \geq a^2\}$, e usando il cambio di coordinate $x = \rho \cos \vartheta$, $y = \rho \sin \vartheta$, per cui $D_{\rho\vartheta}^a = \{(\rho, \vartheta) \in \mathbb{R}^2 : a \leq \rho \leq 2 \cos \vartheta, \vartheta \in [\frac{\pi}{4}, \frac{\pi}{2}]\}$, si ha $\iint_D \frac{1}{\sqrt{x^2+y^2}} dx dy = \lim_{a \rightarrow 0^+} \iint_{D^a} \frac{1}{\sqrt{x^2+y^2}} dx dy = \lim_{a \rightarrow 0^+} \iint_{D_{\rho\vartheta}^a} d\rho d\vartheta = \lim_{a \rightarrow 0^+} \int_{\pi/4}^{\pi/2} \left(\int_a^{2 \cos \vartheta} d\rho \right) d\vartheta = \lim_{a \rightarrow 0^+} \int_{\pi/4}^{\pi/2} (2 \cos \vartheta - a) d\vartheta = \lim_{a \rightarrow 0^+} \left[2 \sin \vartheta - a\vartheta \right]_{\pi/4}^{\pi/2} = \lim_{a \rightarrow 0^+} \left(2 - \sqrt{2} - \frac{\pi}{4} a \right) = 2 - \sqrt{2}$.

(7) Posto $D^a = \{(x, y) \in \mathbb{R}^2 : x^2 + (y - 3)^2 \leq 9, x^2 + y^2 \geq a^2\}$, e usando il cambio di coordinate $x = \rho \cos \vartheta$, $y = \rho \sin \vartheta$, per cui $D_{\rho\vartheta}^a = \{(\rho, \vartheta) \in \mathbb{R}^2 : a \leq \rho \leq 6 \sin \vartheta, \vartheta \in [0, \pi]\}$, si ha $\iint_D \frac{x^2}{x^2+y^2} dx dy = \lim_{a \rightarrow 0^+} \iint_{D^a} \frac{x^2}{x^2+y^2} dx dy = \lim_{a \rightarrow 0^+} \iint_{D_{\rho\vartheta}^a} \rho \cos^2 \vartheta d\rho d\vartheta = \lim_{a \rightarrow 0^+} \int_0^\pi \left(\int_a^{6 \sin \vartheta} \rho \cos^2 \vartheta d\rho \right) d\vartheta = \lim_{a \rightarrow 0^+} \int_0^\pi \cos^2 \vartheta \cdot \left[\frac{1}{2} \rho^2 \right]_a^{6 \sin \vartheta} d\vartheta = \frac{1}{2} \lim_{a \rightarrow 0^+} \int_0^\pi (36 \sin^2 \vartheta - a^2) \cos^2 \vartheta d\vartheta = \frac{1}{2} \lim_{a \rightarrow 0^+} \int_0^\pi (9 \sin^2 2\vartheta - a^2 \cos^2 \vartheta) d\vartheta = \frac{1}{4} \lim_{a \rightarrow 0^+} [(9 - a^2)\vartheta - 9 \cos 4\vartheta - a^2 \cos 2\vartheta]_0^\pi = \frac{1}{4} \lim_{a \rightarrow 0^+} (9 - a^2)\pi = \frac{9}{4} \pi$.

(8) Posto $D^a = \{(x, y) \in \mathbb{R}^2 : x^2 + (y - 3)^2 \leq 9, x^2 + y^2 \geq a^2\}$, e usando il cambio di coordinate $x = \rho \cos \vartheta$, $y = \rho \sin \vartheta$, per cui $D_{\rho\vartheta}^a = \{(\rho, \vartheta) \in \mathbb{R}^2 : a \leq \rho \leq 6 \sin \vartheta, \vartheta \in [0, \pi]\}$, si ha $\iint_D \frac{|xy|e^{x^2+y^2}}{x^2+y^2} dx dy = \lim_{a \rightarrow 0^+} \iint_{D^a} \frac{|xy|e^{x^2+y^2}}{x^2+y^2} dx dy = \lim_{a \rightarrow 0^+} \iint_{D_{\rho\vartheta}^a} \rho e^{\rho^2} |\cos \vartheta \sin \vartheta| d\rho d\vartheta = 2 \lim_{a \rightarrow 0^+} \int_0^{\pi/2} \sin \vartheta \cos \vartheta \left(\int_a^{6 \sin \vartheta} \rho e^{\rho^2} d\rho \right) d\vartheta = 2 \lim_{a \rightarrow 0^+} \int_0^{\pi/2} \sin \vartheta \cos \vartheta \left[\frac{1}{2} e^{\rho^2} \right]_a^{6 \sin \vartheta} d\vartheta = \lim_{a \rightarrow 0^+} \int_0^{\pi/2} (e^{36 \sin^2 \vartheta} - e^{a^2}) \sin \vartheta \cos \vartheta d\vartheta = \lim_{a \rightarrow 0^+} \left[\frac{1}{72} e^{36 \sin^2 \vartheta} - \frac{1}{2} a^2 \sin^2 \vartheta \right]_0^{\pi/2} = \frac{1}{72} (e^{36} - 37)$.

(9) Posto $D^a = \{(x, y) \in \mathbb{R}^2 : x^2 + (y - 1)^2 \geq 1, a^2 \leq x^2 + y^2 \leq 4, |y| \geq |x| \operatorname{tg} a\}$, e usando il cambio di coordinate $x = \rho \cos \vartheta$, $y = \rho \sin \vartheta$, per cui $D_{\rho\vartheta}^a = \{(\rho, \vartheta) \in \mathbb{R}^2 : \max\{a, 2 \sin \vartheta\} \leq \rho \leq 2, \vartheta \in [a, \pi - a]\} \cup \{(\rho, \vartheta) \in \mathbb{R}^2 : a \leq \rho \leq 2, \vartheta \in [\pi + a, 2\pi - a]\}$, si ha $\iint_D \sqrt{\frac{x^2}{|y|}} dx dy = \lim_{a \rightarrow 0^+} \iint_{D^a} \sqrt{\frac{x^2}{|y|}} dx dy = \lim_{a \rightarrow 0^+} \iint_{D_{\rho\vartheta}^a} \frac{\rho |\cos \vartheta|}{\sqrt{|\rho \sin \vartheta|}} \rho d\rho d\vartheta = 2 \lim_{a \rightarrow 0^+} \int_{-\pi/2}^{-a} \frac{\cos \vartheta}{\sqrt{-\sin \vartheta}} \left(\int_a^2 \rho^3 d\rho \right) d\vartheta + 2 \lim_{a \rightarrow 0^+} \int_a^{\pi/2} \frac{\cos \vartheta}{\sqrt{\sin \vartheta}} \left(\int_{2 \sin \vartheta}^2 \rho^3 d\rho \right) d\vartheta = 2 \lim_{a \rightarrow 0^+} \int_{-\pi/2}^{-a} \frac{\cos \vartheta}{\sqrt{-\sin \vartheta}} \left[\frac{2}{5} \rho^{5/2} \right]_a^2 d\vartheta + 2 \lim_{a \rightarrow 0^+} \int_a^{\pi/2} \frac{\cos \vartheta}{\sqrt{\sin \vartheta}} \left[\frac{2}{5} \rho^{5/2} \right]_{2 \sin \vartheta}^2 d\vartheta = \frac{4}{5} \lim_{a \rightarrow 0^+} \int_{-\pi/2}^{-a} \frac{\cos \vartheta}{\sqrt{-\sin \vartheta}} (4\sqrt{2} - a^{5/2}) d\vartheta + \frac{16\sqrt{2}}{5} \lim_{a \rightarrow 0^+} \int_a^{\pi/2} \frac{\cos \vartheta}{\sqrt{\sin \vartheta}} (1 - (\sin \vartheta)^{5/2}) d\vartheta \stackrel{(a)}{=} \frac{4}{5} \lim_{a \rightarrow 0^+} (4\sqrt{2} - a^{5/2}) \int_{\sin a}^1 \frac{1}{\sqrt{t}} dt + \frac{16\sqrt{2}}{5} \lim_{a \rightarrow 0^+} \int_{\sin a}^1 \frac{1}{\sqrt{t}} (1 - t^{5/2}) dt = \frac{4}{5} \lim_{a \rightarrow 0^+} (4\sqrt{2} - a^{5/2}) 2(1 - \sqrt{\sin a}) + \frac{16\sqrt{2}}{5} \lim_{a \rightarrow 0^+} [2\sqrt{t} - \frac{1}{3} t^3]_{\sin a}^1 = \frac{32\sqrt{2}}{5} + \frac{16\sqrt{2}}{5} \lim_{a \rightarrow 0^+} \left(\frac{5}{3} - 2\sqrt{\sin a} + \frac{1}{3} \sin^3 a \right) = \frac{176\sqrt{2}}{15}$, dove in (a) si sono usati i cambi di variabile $t = -\sin \vartheta$ e $t = \sin \vartheta$ nei due integrali.

(10) Posto $D^a = \{(x, y) \in \mathbb{R}^2 : a^2 \leq \frac{1}{16} x^2 + \frac{1}{4} y^2 \leq 1, 2|y| \geq |x| \operatorname{tg} a\}$, e usando il cambio di coordinate $x = 4\rho \cos \vartheta$, $y = 2\rho \sin \vartheta$, per cui $D_{\rho\vartheta}^a = \{(\rho, \vartheta) \in \mathbb{R}^2 : a \leq \rho \leq 1, \vartheta \in [a, \pi - a] \cup [\pi + a, 2\pi - a]\}$, si ha $\iint_D \left| \frac{x}{y} \right| dx dy = \lim_{a \rightarrow 0^+} \iint_{D^a} \left| \frac{x}{y} \right| dx dy = \lim_{a \rightarrow 0^+} \iint_{D_{\rho\vartheta}^a} \left| \frac{4\rho \cos \vartheta}{2\rho \sin \vartheta} \right| 8\rho d\rho d\vartheta = 64 \lim_{a \rightarrow 0^+} \int_a^{\pi/2} \frac{\cos \vartheta}{\sin \vartheta} d\vartheta \cdot \int_a^1 \rho d\rho = 32 \lim_{a \rightarrow 0^+} [\log \sin \vartheta]_a^{\pi/2} \cdot [\rho^2]_a^1 = -32 \lim_{a \rightarrow 0^+} (1 - a^2) \log(\sin a) = +\infty$.

(11) Posto $D^a = \{(x, y) \in \mathbb{R}^2 : x \in [0, 1], x - 1 \leq y \leq x - a\}$, si ha $\iint_D e^{\frac{x+y}{x-y}} dx dy = \lim_{a \rightarrow 0^+} \iint_{D^a} e^{\frac{x+y}{x-y}} dx dy \stackrel{(a)}{=} \frac{1}{2} \lim_{a \rightarrow 0^+} \iint_{D_{st}^a} e^{\frac{s}{t}} ds dt = \frac{1}{2} \lim_{a \rightarrow 0^+} \int_a^1 \left(\int_{-t}^t e^{\frac{s}{t}} ds \right) dt = \frac{1}{2} \lim_{a \rightarrow 0^+} \int_a^1 [te^{\frac{s}{t}}]_{s=-t}^{s=t} dt = \frac{1}{2} \lim_{a \rightarrow 0^+} \int_a^1 (e - \frac{1}{e}) t dt = \frac{1}{4} (e - \frac{1}{e}) \lim_{a \rightarrow 0^+} [t^2]_a^1 = \frac{1}{4} (e - \frac{1}{e}) \lim_{a \rightarrow 0^+} (1 - a^2) = \frac{1}{4} (e - \frac{1}{e})$, dove in (a) si è usato il cambio di variabile $\begin{cases} x + y = s \\ x - y = t \end{cases} \iff \begin{cases} x = \frac{s+t}{2} \\ y = \frac{s-t}{2} \end{cases}$ e $D_{st}^a = \{(s, t) \in \mathbb{R}^2 : t \in [a, 1], -t \leq s \leq t\}$.

(12) Posto $D^a = \{(x, y) \in \mathbb{R}^2 : x \in [a, 1], 0 \leq y \leq x - a\}$, si ha $\iint_D \log \frac{x+y}{x-y} dx dy = \lim_{a \rightarrow 0^+} \iint_{D^a} \log \frac{x+y}{x-y} dx dy \stackrel{(a)}{=} \frac{1}{2} \lim_{a \rightarrow 0^+} \iint_{D_{st}^a} \log \frac{s}{t} ds dt = \frac{1}{2} \lim_{a \rightarrow 0^+} \int_a^1 \left(\int_t^{2-t} (\log s - \log t) ds \right) dt = \frac{1}{2} \lim_{a \rightarrow 0^+} \int_a^1 [s(\log s - 1) - s \log t]_{s=t}^{s=2-t} dt = \frac{1}{2} \lim_{a \rightarrow 0^+} \int_a^1 ((2-t)(\log(2-t) - \log t) + 2(t-1)) dt = \frac{1}{2} \lim_{a \rightarrow 0^+} [(2t - \frac{1}{2} t^2)(\log(2-t) - \log t) + (t-1)^2]_a^1 -$

$$\frac{1}{2} \lim_{a \rightarrow 0^+} \int_a^1 (2t - \frac{1}{2} t^2) (\frac{1}{t-2} - \frac{1}{t}) dt = \frac{1}{2} \lim_{a \rightarrow 0^+} ((2a - \frac{1}{2} a^2) (\log a - \log(2-a)) - (a-1)^2) + \frac{1}{2} \lim_{a \rightarrow 0^+} \int_a^1 (1 - \frac{2}{t-2}) dt = -\frac{1}{2} + \frac{1}{2} \lim_{a \rightarrow 0^+} [t - 2 \log(2-t)]_a^1 = -\frac{1}{2} + \frac{1}{2} \lim_{a \rightarrow 0^+} (1 - a + 2 \log(2-a)) = \log 2, \text{ dove in (a) si}$$

è usato il cambio di variabile $\begin{cases} x+y=s \\ x-y=t \end{cases} \iff \begin{cases} x = \frac{s+t}{2} \\ y = \frac{s-t}{2} \end{cases}$ e $D_{st}^a = \{(s, t) \in \mathbb{R}^2 : t \in [a, 1], t \leq s \leq 2-t\}$.

(13) Posto $D^- = \{(x, y) \in \mathbb{R}^2 : y \in [-2, 0], -2 \leq x \leq y\}$, $D^+ = \{(x, y) \in \mathbb{R}^2 : y \in [0, 2], -2 \leq x \leq y\}$, $D^{-,b} = \{(x, y) \in \mathbb{R}^2 : y \in [-2, -b], -2 \leq x \leq y\}$, $D^{+,a} = \{(x, y) \in \mathbb{R}^2 : y \in [a, 2], -2 \leq x \leq y\}$, si ha $D = D^- \cup D^+$ e $\iint_D \frac{x}{\sqrt{|y|}} dx dy = \iint_{D^-} \frac{x}{\sqrt{|y|}} dx dy + \iint_{D^+} \frac{x}{\sqrt{|y|}} dx dy = \lim_{b \rightarrow 0^+} \iint_{D^{-,b}} \frac{x}{\sqrt{|y|}} dx dy + \lim_{a \rightarrow 0^+} \iint_{D^{+,a}} \frac{x}{\sqrt{|y|}} dx dy = \lim_{b \rightarrow 0^+} \int_{-2}^{-b} (\int_{-2}^y \frac{x}{\sqrt{|y|}} dx) dy + \lim_{a \rightarrow 0^+} \int_a^2 (\int_{-2}^y \frac{x}{\sqrt{|y|}} dx) dy = \lim_{b \rightarrow 0^+} \int_{-2}^{-b} \frac{1}{2\sqrt{|y|}} [x^2]_{-2}^y dy + \lim_{a \rightarrow 0^+} \int_a^2 \frac{1}{2\sqrt{|y|}} [x^2]_{-2}^y dy = \lim_{b \rightarrow 0^+} \int_b^2 \frac{t^2-4}{2\sqrt{t}} dt + \lim_{a \rightarrow 0^+} \int_a^2 \frac{y^2-4}{2\sqrt{y}} dy = \lim_{b \rightarrow 0^+} [\frac{2}{5} t^{5/2} - 8\sqrt{t}]_b^2 + \lim_{a \rightarrow 0^+} [\frac{2}{5} y^{5/2} - 8\sqrt{y}]_a^2 = \lim_{b \rightarrow 0^+} (-\frac{32\sqrt{2}}{5} - \frac{2}{5} b^{5/2} + 8\sqrt{b}) + \lim_{a \rightarrow 0^+} (-\frac{32\sqrt{2}}{5} - \frac{2}{5} a^{5/2} + 8\sqrt{a}) = -\frac{64\sqrt{2}}{5}$.

(14) Posto $D^a = \{(x, y) \in \mathbb{R}^2 : x \in [0, a], 0 \leq y \leq e^{-x}\}$, si ha $\iint_D xy dx dy = \lim_{a \rightarrow +\infty} \iint_{D^a} xy dx dy = \lim_{a \rightarrow +\infty} \int_0^a x (\int_0^{e^{-x}} y dy) dx = \frac{1}{2} \lim_{a \rightarrow +\infty} \int_0^a x [y^2]_{y=0}^{y=e^{-x}} dx = \frac{1}{2} \lim_{a \rightarrow +\infty} \int_0^a x e^{-2x} dx = \frac{1}{2} \lim_{a \rightarrow +\infty} [-\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x}]_0^a = \frac{1}{2} \lim_{a \rightarrow +\infty} (\frac{1}{4} - \frac{a}{2} e^{-2a} - \frac{1}{4} e^{-2a}) = \frac{1}{8}$.

(15) Posto $D^a = \{(x, y) \in \mathbb{R}^2 : x, y \in [-a, a]\}$, si ha $\iint_D \frac{x^2 y^2}{(1+x^6)(1+y^6)} dx dy = \lim_{a \rightarrow +\infty} \iint_{D^a} \frac{x^2 y^2}{(1+x^6)(1+y^6)} dx dy = \lim_{a \rightarrow +\infty} \int_{-a}^a \frac{x^2}{1+x^6} dx \cdot \int_{-a}^a \frac{y^2}{1+y^6} dy = \lim_{a \rightarrow +\infty} (2 \int_0^a \frac{x^2}{1+x^6} dx)^2 \stackrel{(a)}{=} \frac{4}{9} \lim_{a \rightarrow +\infty} (\int_0^{a^3} \frac{dt}{1+t^2})^2 = \frac{4}{9} \lim_{a \rightarrow +\infty} (\arctg(a^3))^2 = \frac{\pi^2}{9}$, dove in (a) si è usato il cambio di variabile $t = x^3$.

(16) Posto $D^a = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0, x^2 + y^2 \leq a^2\}$, si ha $\iint_D \frac{xdy}{(2+x^2+y^2)\sqrt{x^2+y^2}} = \lim_{a \rightarrow +\infty} \iint_{D^a} \frac{xdy}{(2+x^2+y^2)\sqrt{x^2+y^2}} = \lim_{a \rightarrow +\infty} \int_0^{\pi/2} (\int_0^a \frac{1}{(2+\varrho^2)\varrho} \varrho d\varrho) d\vartheta = \frac{\pi}{2} \lim_{a \rightarrow +\infty} [\frac{1}{\sqrt{2}} \arctg \frac{\varrho}{\sqrt{2}}]_0^a = \frac{\pi}{2\sqrt{2}} \lim_{a \rightarrow +\infty} \arctg \frac{a}{\sqrt{2}} = \frac{\pi^2}{4\sqrt{2}}$.

(17) Posto $D^a = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq a^2\}$, si ha $\iint_D \frac{x^2 y^2 e^{-x^2-y^2}}{x^2+y^2} dx dy = \lim_{a \rightarrow +\infty} \iint_{D^a} \frac{x^2 y^2 e^{-x^2-y^2}}{x^2+y^2} dx dy = \lim_{a \rightarrow +\infty} \int_0^{2\pi} \cos^2 \vartheta \sin^2 \vartheta d\vartheta \cdot \int_0^a \varrho^3 e^{-\varrho^2} d\varrho \stackrel{(a)}{=} \lim_{a \rightarrow +\infty} \int_0^{2\pi} \frac{1}{8} (1 - \cos 4\vartheta) d\vartheta \cdot \int_0^{a^2} \frac{1}{2} t e^{-t} dt = \frac{1}{16} \lim_{a \rightarrow +\infty} [\vartheta - \frac{1}{4} \sin 4\vartheta]_0^{2\pi} \cdot [- (t+1)e^{-t}]_0^{a^2} = \frac{\pi}{8} \lim_{a \rightarrow +\infty} (1 - (a^2+1)e^{-a^2}) = \frac{\pi}{8}$, dove in (a) si è usato il cambio di variabile $t = \varrho^2$.

(18) La funzione integranda è positiva, per cui si può calcolare l'integrale usando un'esaustione qualsiasi. Posto, per ogni $a \in (0, 1)$, $D^a = \{(x, y, z) \in \mathbb{R}^3 : a \leq x \leq 1, |y| \leq x, 1 \leq z \leq \frac{4}{4x-y^2}\}$, si ha $\iiint_D \frac{dx dy dz}{x z^2} = \lim_{a \rightarrow 0^+} \int_a^1 \frac{dx}{x} \int_{-x}^x dy \int_1^{4/(4x-y^2)} \frac{dz}{z^2} = \lim_{a \rightarrow 0^+} \int_a^1 \frac{dx}{x} \int_{-x}^x [-\frac{1}{z}]_1^{4/(4x-y^2)} dy = \lim_{a \rightarrow 0^+} \int_a^1 \frac{dx}{x} \int_{-x}^x (1 - x + \frac{y^2}{4}) dy = \lim_{a \rightarrow 0^+} \int_a^1 [y - xy + \frac{y^3}{12}]_{-x}^x \frac{dx}{x} = 2 \lim_{a \rightarrow 0^+} \int_a^1 (x - x^2 + \frac{x^3}{12}) \frac{dx}{x} = 2 \lim_{a \rightarrow 0^+} [x - \frac{x^2}{2} + \frac{x^3}{36}]_a^1 = \frac{19}{18}$.

(19) La funzione integranda è positiva, per cui si può calcolare l'integrale usando un'esaustione qualsiasi. Usando il cambiamento di coordinate sferiche $x = \varrho \cos \vartheta \sin \varphi$, $y = \frac{1}{2} \varrho \sin \vartheta \sin \varphi$, $z = \varrho \cos \varphi$ si ottiene $D_{\varrho\vartheta\varphi} = \{(\varrho, \vartheta, \varphi) \in \mathbb{R}^3 : 0 \leq \vartheta \leq 2\pi, 0 \leq \varrho \leq 1, 0 \leq \varphi \leq \frac{\pi}{2}\}$. Posto, per ogni

$a \in (0, 1)$, $D_{\varrho\vartheta\varphi}^a = \{(\varrho, \vartheta, \varphi) \in \mathbb{R}^3 : 0 \leq \vartheta \leq 2\pi, 0 \leq \varrho \leq 1 - a, 0 \leq \varphi \leq \frac{\pi}{2}\}$, si ha

$$\begin{aligned} \iiint_D \frac{z \, dx \, dy \, dz}{\sqrt{1 - x^2 - 4y^2 - z^2}} &= \lim_{a \rightarrow 0^+} \iiint_{D_{\varrho\vartheta\varphi}^a} \frac{\varrho \cos \varphi}{\sqrt{1 - \varrho^2}} \cdot \frac{1}{2} \varrho^2 \sin \varphi \, d\varrho \, d\vartheta \, d\varphi \\ &= \frac{1}{2} \lim_{a \rightarrow 0^+} \int_0^{\pi/2} \sin \varphi \cos \varphi \, d\varphi \int_0^{2\pi} d\vartheta \int_0^{1-a} \frac{\varrho^3 \, d\varrho}{\sqrt{1 - \varrho^2}} \stackrel{(a)}{=} \pi \left[\frac{1}{2} \sin^2 \varphi \right]_0^{\pi/2} \lim_{a \rightarrow 0^+} \int_{2a-a^2}^1 \frac{1}{2} \left(\frac{1}{\sqrt{t}} - \sqrt{t} \right) dt \\ &= \frac{\pi}{4} \lim_{a \rightarrow 0^+} \left[2\sqrt{t} - \frac{2}{3} t^{3/2} \right]_{2a-a^2}^1 = \frac{\pi}{3}, \end{aligned}$$

dove in (a) si è usata la sostituzione $1 - \varrho^2 = t \implies -2\varrho \, d\varrho = dt$.

(20) La funzione integranda è positiva, per cui si può calcolare l'integrale usando un'esaustione qualsiasi. Passando in coordinate cilindriche, si ha $D_{\varrho\vartheta z} = \{(\varrho, \vartheta, z) \in \mathbb{R}^3 : \varrho^2 \leq z, 0 \leq \vartheta \leq 2\pi\}$. Posto, per ogni $a > 0$, $D_{\varrho\vartheta z}^a := \{(\varrho, \vartheta, z) \in \mathbb{R}^3 : \varrho^2 \leq z \leq a, 0 \leq \vartheta \leq 2\pi\}$, si ha $\iiint_D \frac{x^2}{1+z^6} \, dx \, dy \, dz = \lim_{a \rightarrow +\infty} \iiint_{D_{\varrho\vartheta z}^a} \frac{\varrho^2 \cos^2 \vartheta}{1+z^6} \varrho \, d\varrho \, d\vartheta = \lim_{a \rightarrow +\infty} \int_0^a \frac{dz}{1+z^6} \int_0^{2\pi} \cos^2 \vartheta \, d\vartheta \int_0^{\sqrt{z}} \varrho^3 \, d\varrho = \lim_{a \rightarrow +\infty} \left[\frac{\vartheta}{2} + \frac{\sin 2\vartheta}{4} \right]_0^{2\pi} \int_0^a \left[\frac{\varrho^4}{4} \right]_0^{\sqrt{z}} \frac{dz}{1+z^6} = \lim_{a \rightarrow +\infty} \frac{\pi}{4} \int_0^a \frac{z^2 \, dz}{1+z^6} = \frac{\pi}{4} \lim_{a \rightarrow +\infty} \left[\frac{1}{3} \arctg(z^3) \right]_0^a = \frac{\pi^2}{12}$.

(21) La funzione integranda è positiva, per cui si può calcolare l'integrale usando un'esaustione qualsiasi. Usando il cambiamento di coordinate sferiche $x = \varrho \cos \vartheta \sin \varphi$, $y = \varrho \sin \vartheta \sin \varphi$, $z = \varrho \cos \varphi$ si ottiene $D_{\varrho\vartheta\varphi} = \{(\varrho, \vartheta, \varphi) \in \mathbb{R}^3 : 0 \leq \vartheta \leq 2\pi, 0 \leq \varphi \leq \pi, \varrho \geq 1\}$. Posto, per ogni $a > 0$, $D_{\varrho\vartheta\varphi}^a = \{(\varrho, \vartheta, \varphi) \in \mathbb{R}^3 : 0 \leq \vartheta \leq 2\pi, 0 \leq \varphi \leq \pi, 1 \leq \varrho \leq a\}$, si ha

$$\begin{aligned} \iiint_D \frac{x^2 \, dx \, dy \, dz}{(x^2 + y^2 + z^2)^4} &= \lim_{a \rightarrow +\infty} \iiint_{D_{\varrho\vartheta\varphi}^a} \frac{\varrho^2 \cos^2 \vartheta \sin^2 \varphi}{\varrho^8} \varrho^2 \sin \varphi \, d\varrho \, d\vartheta \, d\varphi \\ &= \lim_{a \rightarrow +\infty} \int_0^\pi (1 - \cos^2 \varphi) \sin \varphi \, d\varphi \int_0^{2\pi} \frac{1 + \cos 2\vartheta}{2} \, d\vartheta \int_1^a \frac{d\varrho}{\varrho^4} \\ &= \left[-\cos \varphi + \frac{\cos^3 \varphi}{3} \right]_0^\pi \cdot \left[\frac{\vartheta}{2} + \frac{\sin 2\vartheta}{4} \right]_0^{2\pi} \lim_{a \rightarrow +\infty} \left[-\frac{1}{3\varrho^3} \right]_1^a = \frac{4\pi}{9}. \end{aligned}$$

□